

FINDING EIGENVALUES AND CORRESPONDING EIGENVECTORS THROUGH APPLYING (F.E.M) TO ELLIPTIC (P.D.E) USING DIFFERENT METHODS

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Abstract:-

In this paper the power method and the symmetric power method are applied to the matrix H which was obtained from a linear system $Hx = b$. This system was derived from an elliptic partial differential equation by adopting the finite element method in order to find eigenvalues and the corresponding eigenvectors. These methods are applied to two examples. The different of the two examples are compared to find out the differences between them. The paper has shown that the use of symmetric power method leads to a large eigenvalue if compared to the use of the power method.

Keywords:-*Power method, Symmetric power method, Eigenvalues, Eigenvectors. Elliptic partial differential equations, Finite element methods.*

INTRODUCTION:

Let Ω be a bounded domain in R^d , with boundary $\partial\Omega$. We assume that f and g are continuous on Ω . So there exists unique solution

$$-\Delta u = f(x, y) \text{ for } (x, y) \in \Omega \text{ and}$$

$$u(x, y) = g(x, y) \text{ in } \partial\Omega$$

$$R = \{(x, y): a < x < b, c < y < d\}.$$

Discretization of elliptic partial differential equations by finite element method leads to a linear system of equations of the form $Hx = b$.

Here $H \in R^{n \times n}$ is symmetric matrix, $b \in R^n$ and for the solution $x \in R^n$ is obtained.

In this paper some well-known methods are considered to find the large eigenvalues and corresponding eigenvectors to the matrix H which was derived from the linear system $Hx = b$ such as power method and symmetric power method.

[1] Power method.

Assume that the $n \times n$ matrix H has n eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ with an associated collection of eigenvectors $v^{(1)}, v^{(2)}, v^{(3)}, \dots, v^{(n)}$ that is linear independent. Assume that λ_1 is largest of the matrix H with

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n| \geq 0$$

At the beginning we select a unit vector $x^{(0)}$ with

$$x(0) = 1 = \|x(0)\|_\infty$$

Let

$$y(1) = Hx(0)$$

So, we define

$u(1) = y_p(1)$ $u^{(1)}$ is the largest approximate eigenvalues for $Hx^{(0)}$

Let p be the smallest integer, $1 \leq p \leq n$ such that

$$|y_p(1)| = \|y(1)\|_\infty$$

So, we define eigenvector $x^{(1)}$ corresponding to the eigenvalue by

$$x^{(1)} = \frac{y^{(1)}}{y_p^{(1)}} = \frac{Hx^{(0)}}{y_p^{(1)}}$$

Then

$$x^{(1)} = 1 = \|x^{(1)}\|_\infty$$

So, we define

$$y^{(2)} = Hx^{(1)} = \frac{H^2 x^{(0)}}{y_p^{(1)}}$$

Now we find

$$u^{(2)} = y_p^{(2)}$$

With

$$|y_p^{(2)}| = \|y^{(2)}\|_\infty$$

So, we define

$$y^{(2)} = Hx^{(1)} = \frac{H^2 x^{(0)}}{y_p^{(1)}}$$

The approximated eigenvector corresponding to $u(2)$ is defined as:

$$x^{(2)} = \frac{y^{(2)}}{y_p^{(2)}} = \frac{Hx^{(1)}}{y_p^{(2)}} = \frac{H^2 x^{(0)}}{y_p^{(2)} y_p^{(1)}}$$

With the similar manner we get

$$y^{(m)} = Hx^{(m-1)},$$

$$\mu^{(m)} = y_p^{(m)},$$

$$x^{(m)} = \frac{y^{(m)}}{y_p^{(m)}} = \frac{H^{(m)} x^{(0)}}{\prod_{k=1}^m y_p^{(k)}}$$

With

$$|y_p^{(m)}| = \|y^{(m)}\|_\infty$$

[2] Symmetric power method.

Through this method we will find the largest eigenvalues and corresponding eigenvectors to the square symmetric matrix H , we choose initial approximated vector $x^{(0)}$ with

$$\|x^{(0)}\|_2 = 1,$$

Let,

$$y^{(m)} = Hx^{(m-1)},$$

So we define,

$\mu^{(m)} = (x^{(m-1)})^T y^{(m)}$,
 $\mu^{(m)}$ is the approximate solution of eigenvalue for $\forall m = 1, 2, \dots, n$ With corresponding eigenvector.

$$x^{(m)} = \frac{y^{(m)}}{\|y^{(m)}\|_2} = \frac{Hx^{(m-1)}}{\|Hx^{(m-1)}\|_2}.$$

[3] Aitken method.

Aitken method can be used to accelerate the convergence of any sequence that linearly convergent.

Assume that $\{u_n\}_{n=0}^{\infty}$ is a linearly convergent sequence with limit u . Then the sequence $\{\tilde{u}_n\}_{n=0}^{\infty}$ is convergent more rapidly to u .

Assume that

$$\begin{aligned} \frac{u_{n+1} - u}{u_n - u} &\cong \frac{u_{n+2} - u}{u_{n+1} - u}, \\ (u_{n+1} - u)^2 &\cong (u_{n+2} - u)(u_n - u), \\ u_{n+1}^2 - 2u_{n+1}u + u^2 &\cong u_{n+2}u_n - (u_n + u_{n+2})u + u^2, \\ u &\cong \frac{u_{n+2}u_n - u_{n+1}^2}{u_{n+2} - 2u_{n+1} - u_n}. \end{aligned}$$

Adding and subtracting u_n^2 , $2u_n u_{n+1}$ in the numerator

$$\begin{aligned} u &\cong \frac{u_{n+2}u_n - 2u_n u_{n+1} + u_n^2 - u_{n+1}^2 + 2u_n u_{n+1} - u_n^2}{u_{n+2} - 2u_{n+1} - u_n}, \\ u &\cong \frac{u_n(u_{n+2} - 2u_{n+1} + u_n) - (u_{n+1}^2 - 2u_n u_{n+1} + u_n^2)}{u_{n+2} - 2u_{n+1} - u_n}, \\ u &\cong u_n - \frac{(u_{n+1} - u_n)^2}{u_{n+2} - 2u_{n+1} - u_n} \end{aligned}$$

Aitken method is based on the assumption that the sequence $\{\tilde{u}_n\}_{n=0}^{\infty}$ defined by

$$\tilde{u}_n = u_n - \frac{(u_{n+1} - u_n)^2}{u_{n+2} - 2u_{n+1} - u_n}$$

Convergence more rapidly to u than the original sequence $\{u_n\}_{n=0}^{\infty}$.

[4] Results and Applications

For reaching the approximate results for a large eigenvalue, we will study these two examples for power method and symmetric power method.

Example 1

Consider the problem in the given domain with boundary conditions

$$\begin{aligned} -\Delta u &= 2 - (x^2 + y^2) & 0 \leq x, y \leq 1 \\ u(x, y) &= 0 & \text{in boundary} \end{aligned}$$

Example 2

Consider the problem in the given domain with boundary conditions

$$\begin{aligned} -\Delta u &= 2 - (x^2 + y^2) & 0 \leq x, y \leq 1; \\ u(x, y) &= 0 & \text{when } x = 1, y = 1; \\ u(x, y) &= \frac{1}{2}(1 - y^2) & \text{when } x = 0; \\ u(x, y) &= \frac{1}{2}(1 - x^2) & \text{when } y = 0. \end{aligned}$$

Figure 1 The shape of example 1 when $n = 10$

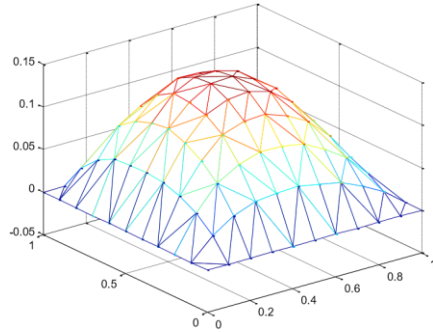


Figure 2 The shape of matrix H of example 1 when $n = 5$

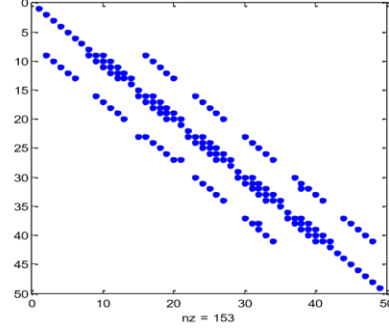


Figure 1 The shape of example 1 when $n = 10$

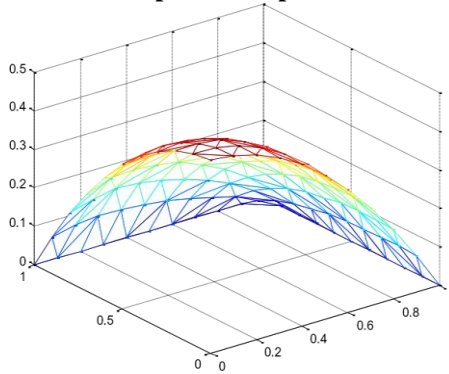
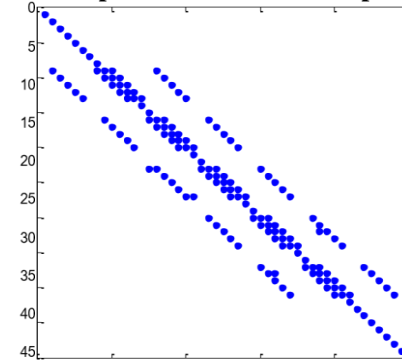


Figure 4 The shape of matrix H of example 1 when $n = 5$



Numerical Results

The results of numerical tests of the power method, symmetric power method and Aitken method for example 1 are given in the following tables:

(i) When the order of the matrix H is 36x36

(ii)

Table 1 eigenvalues using the power method

Iteration m	eigenvalue $\lambda^{(m)}$
1	1.0
2	2.0
⋮	⋮
20	5.2354392
⋮	⋮
40	5.2360696
⋮	⋮
60	5.2371140
⋮	⋮
80	6.2787105
⋮	⋮
100	7.2347309
⋮	⋮
120	7.2360659 ...
⋮	⋮
126	7.2360677

Table 2 eigenvalues using the symmetric power method

Iteration m	eigenvalue $\lambda^{(m)}$
1	20.0000000
2	1.0000000
⋮	⋮
20	5.2360677
⋮	⋮
40	5.2679575
⋮	⋮
60	7.2357719
⋮	⋮
73	7.2360679

In the following step we will apply the Aitken method to the approximated eigenvalues for acceleration the convergence to a large eigenvalue

Table 3 eigenvalues using Aitken method for the power method

Iteration m	Eigenvalue $\hat{\lambda}^{(m)}$
1	0.0000000
2	3.7777778
⋮	⋮
10	5.2400189
⋮	⋮
20	5.2360680
⋮	⋮
30	5.2359710
⋮	⋮
40	5.1714694
⋮	⋮
50	7.2381778
⋮	⋮
60	7.2360712
⋮	⋮
66	7.2360681

Table 4 eigenvalues using Aitken method for the symmetric power method

Iteration m	Eigenvalue $\hat{\lambda}^{(m)}$
1	2.1928251
2	6.9150522
⋮	⋮
10	5.2360813
⋮	⋮
20	5.2360541
⋮	⋮
30	5.2270442
⋮	⋮
40	7.2508583
⋮	⋮
50	7.2360907
⋮	⋮
60	7.2360681

When the order of the matrix is 100x100

Table 5 eigenvalues using the power method

Iteration m	eigenvalue $\lambda^{(m)}$
1	1.0000000
2	2.0000000
⋮	⋮
40	7.0623488
⋮	⋮
80	7.0641763
⋮	⋮
120	7.0642066
⋮	⋮
160	7.0654026
⋮	⋮
200	7.1127712
⋮	⋮
240	7.7478043
⋮	⋮
280	7.7585089
⋮	⋮
320	7.7587643
⋮	⋮
339	7.7587694

Table 6 eigenvalues using the symmetric power method

Iteration m	Eigenvalue $\lambda_1^{(m)}$
1	36.0000000
2	1.0000000
⋮	⋮
40	7.0641660
⋮	⋮
80	7.0643330
⋮	⋮
120	7.2644847
⋮	⋮
160	7.7578266
⋮	⋮
200	7.7587700

By applying the Aitken method to the approximated eigenvalues for acceleration the convergence to a large eigenvalue as follow:

Table 7 eigenvalues using Aitken method for the power method

Iteration m	Eigenvalue $\hat{\lambda}_1^{(m)}$
1	0.0000000
2	3.7777778
⋮	⋮
40	7.0641809
⋮	⋮
80	7.0638345
⋮	⋮
120	8.1422186
⋮	⋮
160	7.7587725
⋮	⋮
167	7.7587710

Table 8 eigenvalues using Aitken method for the symmetric power method

Iteration m	Eigenvalue $\hat{\lambda}_1^{(m)}$
1	2.1208151
2	7.5789474
⋮	⋮
40	7.0641732
⋮	⋮
80	7.0557540
⋮	⋮
120	7.7588549
⋮	⋮
147	7.7587710

After applying the power method, the symmetric power method and Aitken method for example 1 with 7 digits accuracy we have got the following results:

- When the matrix H of order 36x36 we found out the large eigenvalue at the step 126 of power method, in the accelerating approximation by Aitken method the large eigenvalue at the step 66. While the symmetric power method we got the approximate eigenvalue at the step 73, accelerating approximation by Aitken method the large eigenvalue at the step 60.
- When H of order 100x100 we have got the large eigenvalue at the step 339 of power method, , in the accelerating approximation by Aitken method the large eigenvalue at the step 167, While the symmetric power method we got the approximate eigenvalue at the step 200, in the accelerating approximation by Aitken method the large eigenvalue at the step 147.

The results of numerical tests of the power method and symmetric power method for example 2 are given in the following tables:

(i) When the order of the matrix H is 36x36

Table 9 eigenvalues using the power method

Iteration m	Eigenvalue $\lambda^{(m)}$
1	1.0000000
2	2.0000000
⋮	⋮
10	5.0823558
⋮	⋮
20	5.2354392
⋮	⋮
30	5.2360656
⋮	⋮
40	5.2360696
⋮	⋮
50	5.2361092
⋮	⋮
60	5.2371140
⋮	⋮
70	5.2623107
⋮	⋮
80	6.2787105
⋮	⋮
90	7.2026410
⋮	⋮
100	7.2347309
⋮	⋮
110	7.2360153
⋮	⋮
120	7.2360659
⋮	⋮
126	7.2360677

Table 10 eigenvalues using the symmetric power method

Iteration m	Eigenvalue $\lambda^{(m)}$
1	20.0000000
2	1.0000000
⋮	⋮
10	5.2202585
⋮	⋮
20	5.2360677
⋮	⋮
30	5.2361182
⋮	⋮
40	5.2679575
⋮	⋮
50	7.0615529
⋮	⋮
60	7.2357719
⋮	⋮
70	7.2360675
⋮	⋮
72	7.2360679

In the following step we will apply the Aitken method to the approximated eigenvalues for acceleration the convergence to a large eigenvalue

Table 11 eigenvalues using Aitken method for the power method

Iteration m	Eigenvalue $\hat{\lambda}^{(m)}$
1	0.0000000
2	3.7777778
⋮	⋮
10	5.2400189
⋮	⋮
20	5.2360680
⋮	⋮
30	5.2359710
⋮	⋮
40	5.1714694
⋮	⋮
50	7.2381778
⋮	⋮
60	7.2360712
⋮	⋮
65	7.2360681

Table 12 eigenvalues using Aitken method for the symmetric power method

Iteration m	Eigenvalue $\hat{\lambda}^{(m)}$
1	2.1928251
2	6.9150522
⋮	⋮
10	5.2360813
⋮	⋮
20	5.2360541
⋮	⋮
30	5.2270442
⋮	⋮
40	7.2508583
⋮	⋮
50	7.2360907
⋮	⋮
58	7.2360681

(ii) When the order of the matrix H is 100x100

Table 13 eigenvalues using the power method

Iteration m	eigenvalue $\lambda^{(m)}$
1	1.0000000
2	2.0000000
⋮	⋮
40	7.0623488
⋮	⋮
80	7.0641766
⋮	⋮
120	7.0642094
⋮	⋮
160	7.0655227
⋮	⋮
200	7.1171846
⋮	⋮
240	7.7487719
⋮	⋮
280	7.7585323
⋮	⋮
320	7.7587649
⋮	⋮
339	7.7587694

Table 14 eigenvalues using the symmetric power method

Iteration m	eigenvalue $\lambda_1^{(m)}$
1	36.0000000
2	1.0000000
⋮	⋮
40	7.0641660
⋮	⋮
80	7.0643330
⋮	⋮
120	7.2644847
⋮	⋮
160	7.7578266
⋮	⋮
200	7.7587700

By applying the Aitken method to the approximated eigenvalues for acceleration the convergence to a large eigenvalue as follow:

Table 15 eigenvalues using Aitken method for the power method

Iteration m	Eigenvalue $\hat{\lambda}_1^{(m)}$
1	0.0000000
2	3.7777778
⋮	⋮
40	7.0641809
⋮	⋮
80	7.0638345
⋮	⋮
120	7.9352722
⋮	⋮
160	7.7587716
⋮	⋮
164	7.7587710

Table 16 eigenvalues using Aitken method for the symmetric power method

Iteration m	Eigenvalue $\hat{\lambda}_1^{(m)}$
1	2.1208151
2	7.5789474
⋮	⋮
40	7.0641732
⋮	⋮
80	7.0557540
⋮	⋮
120	7.7588549
⋮	⋮
147	7.7587710

After applying the power method, the symmetric power method and Aitken method for example 2 with 8 digits we have got the following results:

- When the matrix H of order 36x36 we found out the large eigenvalue at the step 126 of power method, in the accelerating approximation by Aitken method the large eigenvalue at the step 65. While the symmetric power method we got the approximate eigenvalue at the step 72, accelerating approximation by Aitken method the large eigenvalue at the step 58.
- When the matrix H of order 100x100 we have got the large eigenvalue at the step 339 of power method, , in the accelerating approximation by Aitken method the large eigenvalue at the step 164, While the symmetric power method we got the approximate eigenvalue at the step 200, in the accelerating approximation by Aitken method the large eigenvalue at the step 147.

Conclusion.

In the paper we studied power method, symmetric power method and Aitken method, applying them to the matrix H to find the large eigenvalue. The results reviewed that when the order of the matrix H increase the number of iterations increase. So the convergent to a large eigenvalue by the symmetric power methodis faster than the power method. While the convergent accelerate by using the Aitken methodyields to a better results.

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