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# FINDING EIGENVALUES AND CORRESPONDING EIGENVECTORS THROUGH APPLYING (F.E.M) TO ELLIPTIC (P.D.E) USING DIFFERENT METHODS

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## Abstract:-

*In this paper the power method and the symmetric power method are applied to the matrix $H$ which was otained from a linear system $Hx = b$. This system was derived from an elliptic partial differential equation by adopting the finite element method in order to find eigenvalues and the corresponding eigenvectors. These methods are applied to two examples. The different of the two examples are compared to find out the differences between them. The paper has shown that the use of symmetric power method leads to a large eigenvalue if compared to the use of the power method.*

**Keywords:**-*Power method, Symmetric power method, Eigenvalues, Eigenvectors. Elliptic partial differential equations, Finite element methods.*

**INTRODUCTION**:
Let $\Omega$be a bounded domain in $R^d$, with boundary $\partial\Omega$. We assume that f and g are continuous on $\Omega$. So there exists unique solution
$-\Delta u = f(x, y)\ for\ (x, y) \in \Omega$ and

$$u(x, y) = g(x, y)\ in\ \partial\Omega$$
$$R = \{(x, y): a < x < b, c < y < d\}.$$

Discretization of elliptic partial differential equations by finite element method leads to a linear system of equations of the form $Hx = b$.
Here $H \in R^{nxn}$ is symmetric matrix, $b\in R^n$and for the solution $x\in R^n$ is obtained.
In this paper some well-known methods are considered to find the large eigenvalues and corresponding eigenvectors to the matrix $H$ which was drived from the linear system $Hx = b$ such as power method and symmetric power method.

**[1] Power method.**
Assume that the $nxn$ matrix $H$ has $n$ eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ with an associated collection of eigenvectors $v^{(1)}, v^{(2)}, v^{(3)}, \ldots, v^{(n)}$ that is linear independent. Assume that $\lambda_1$ is largest of the matrix $H$ with

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \geq 0$$

At the beginning we select a unit vector $x^{(0)}$ with

$$x(0) = 1 = \|x(0)\|_{\infty}$$

Let

$$y(1) = Hx(0)$$

So, we define

$u(1) = {}^{y}{}_p(1)\ u^{(1)}$ is the largest approximate eigenvalues for $Hx^{(0)}$
Let $p$ be the smallest integer, $1 \leq p \leq n$ such that

$$|yp(1)| = \|y(1)\|\infty$$

So, we define eigenvector $x^{(1)}$ corresponding to the eigenvalue by

$$x^{(0)} = \frac{y^{(0)}}{y_p^{(1)}} = \frac{Hx^{(0)}}{y_p^{(1)}}$$

Then

$$x^{(1)} = 1 = \left\|x^{(1)}\right\|_{\infty}$$

So, we define

$$y^{(2)} = Hx^{(1)} = \frac{H^2x^{(0)}}{y_p^{(1)}}$$

Now we find

$$u^{(2)} = y_p^{(2)}$$

With

$$\left|y_p^{(2)}\right| = \left\|y^{(2)}\right\|_{\infty}$$

So, we define

$$y^{(2)} = Hx^{(1)} = \frac{H^2x^{(0)}}{y_p^{(1)}}$$

The approximated eigenvector corresponding to $u(2)$is defined as:

$$x^{(2)} = \frac{y^{(2)}}{y_p^{(2)}} = \frac{Hx^{(1)}}{y_p^{(2)}} = \frac{H^2x^{(0)}}{y_p^{(2)}y_p^{(1)}}$$

With the similar manner we get

$$y^{(m)} = Hx^{(m-1)},$$
$$\mu^{(m)} = y_p^{(m)},$$
$$x^{(m)} = \frac{y^{(m)}}{y_p^{(m)}} == \frac{H^{(m)}x^{(0)}}{\prod_{k=1}^{m} y_p^{(k)}}$$

With

$$\left|y_p^{(m)}\right| = \left\|y^{(m)}\right\|_{\infty}$$

**[2] Symmetric power method.**
Through this method we will find the largest eigenvalues and corresponding eigenvectors to the square symmetric matrix $H$, we choose initial approximated vector $x^{(0)}$ with

$$\|x^{(0)}\|_2 = 1,$$

Let,

$$y^{(m)} = Hx^{(m-1)},$$

So we define,

$$\mu^{(m)} = (x^{(m-1)})^T y^m,$$

$\mu^{(m)}$ is the approximate solution of eigenvalue for $\forall \ m = 1,2,\dots,n$ With corresponding eigenvector.

$$x^{(m)} = \frac{y^{(m)}}{\|y^{(m)}\|_2} == \frac{Hx^{(m-1)}}{\|Hx^{(m-1)}\|_2}.$$

**[3] Aitken method.**

Aitken method can be used to accelerate the convergence of any sequence that linearly convergent.
Assume that $\{u_n\}_{n=0}^{\infty}$is a linearly convergent sequence with limit $u$. Then the sequence $\{\tilde{u}_n\}_{n=0}^{\infty}$is convergent more rapidly to $u$.
Assume that

$$\frac{u_{n+1} - u}{u_n - u} \cong \frac{u_{n+2} - u}{u_{n+1} - u},$$

$$(u_{n+1} - u)^2 \cong (u_{n+2} - u)(u_n - u),$$

$${u_{n+1}}^2 - 2u_{n+1}u + u^2 \cong u_{n+2}u_n - (u_n + u_{n+2})u + u^2,$$

$$u \cong \frac{u_{n+2}u_n - {u_{n+1}}^2}{u_{n+2} - 2u_{n+1} - u_n},$$

Adding and subtracting ${u_n}^2$, $2u_n u_{n+1}$ in the numerator

$$u \cong \frac{u_{n+2}u_n - 2u_n u_{n+1} + {u_n}^2 - {u_{n+1}}^2 + 2u_n u_{n+1} - {u_n}^2}{u_{n+2} - 2u_{n+1} - u_n},$$

$$u \cong \frac{u_n(u_{n+2} - 2u_{n+1} + u_n) - ({u_{n+1}}^2 - 2u_n u_{n+1} + {u_n}^2)}{u_{n+2} - 2u_{n+1} - u_n},$$

$$u \cong u_n - \frac{(u_{n+1} - u_n)^2}{u_{n+2} - 2u_{n+1} - u_n}$$

Aitken method is based on the assumption that the sequence $\{\tilde{u}_n\}_{n=0}^{\infty}$defined by

$$\tilde{u} = u_n - \frac{(u_{n+1} - u_n)^2}{u_{n+2} - 2u_{n+1} - u_n}$$

Convergence more rapidly to $u$ than the original sequence $\{u_n\}_{n=0}^{\infty}$.

**[4] Results and Applications**

For reaching the approximate results for a large eigenvalue, we will study these two examples for power method and symmetric power method.

**Example 1**

Consider the problem in the given domain with boundary conditions

$$-\Delta u = 2 - (x^2 + y^2) \qquad 0 \le x, y \le 1$$
$$u(x, y) = 0 \qquad \text{in boundary}$$

**Example 2**

Consider the problem in the given domain with boundary conditions

$$-\Delta u = 2 - (x^2 + y^2) \qquad 0 \le x, y \le 1;$$
$$u(x, y) = 0 \qquad \text{when x = 1 , y = 1;}$$
$$u(x, y) = \frac{1}{2}(1 - y^2) \text{ when x = 0;}$$
$$u(x, y) = \frac{1}{2}(1 - x^2) \text{ when y = 0.}$$

Figure 1 The shape of example 1 when n = 10

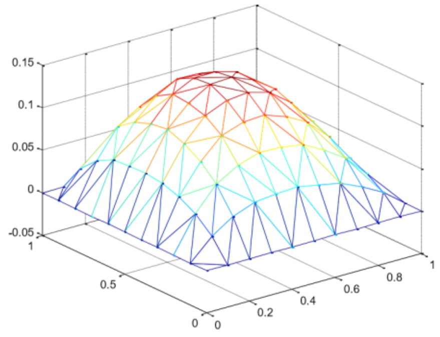


Figure 2 The shape of matrix H of example 1 when n = 5

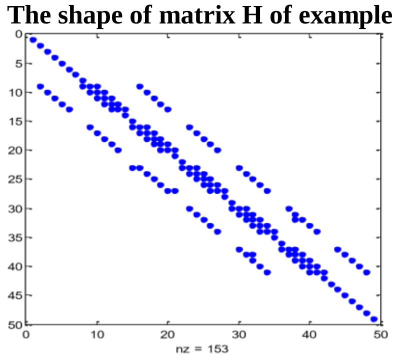



Figure 1 The shape of example 1 when n = 10

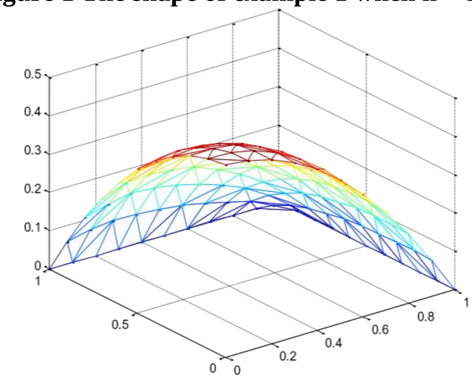
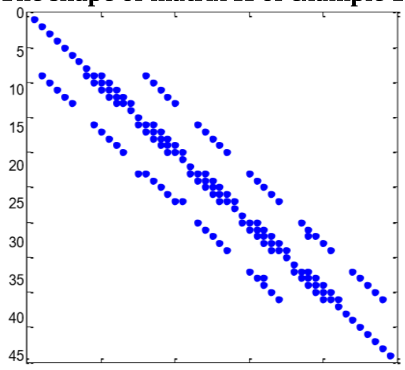


Figure 4 The shape of matrix H of example 1 when n = 5



**Numerical Results**

The results of numerical tests of the power method, symmetric power method and Aitken method for example 1 are given in the following tables:

**(i)** When the order of the matrix H is 36x36

**(ii)**

**Table 1 eigenvalues using the power method**

| Iteration m | eigenvalue $u^{(m)}$ |
|---|---|
| 1 | 1.0 |
| 2 | 2.0 |
| ⋮ | ⋮ |
| 20 | 5.2354392 |
| ⋮ | ⋮ |
| 40 | 5.2360696 |
| ⋮ | ⋮ |
| 60 | 5.2371140 |
| ⋮ | ⋮ |
| 80 | 6.2787105 |
| ⋮ | ⋮ |
| 100 | 7.2347309 |
| ⋮ | ⋮ |
| 120 | 7.2360659 … |
| ⋮ | ⋮ |
| 126 | 7.2360677 |

**Table 2 eigenvalues using thesymmetric power method**

| Iteration m | eigenvalue $u^{(m)}$ |
|---|---|
| 1 | 20.0000000 |
| 2 | 1.0000000 |
| ⋮ | ⋮ |
| 20 | 5.2360677 |
| ⋮ | ⋮ |
| 40 | 5.2679575 |
| ⋮ | ⋮ |
| 60 | 7.2357719 |
| ⋮ | ⋮ |
| 73 | 7.2360679 |

In the following step we will apply the Aitken method to the approximated eigenvalues for acceleration the convergence to a large eigenvalue

**Table 3 eigenvalues using Aitken method for the power method**

| Iteration m | Eigenvalue $\tilde{u}^{(m)}$ |
|---|---|
| 1 | 0.0000000 |
| 2 | 3.7777778 |
| ⋮ | ⋮ |
| 10 | 5.2400189 |
| ⋮ | ⋮ |
| 20 | 5.2360680 |
| ⋮ | ⋮ |
| 30 | 5.2359710 |
| ⋮ | ⋮ |
| 40 | 5.1714694 |
| ⋮ | ⋮ |
| 50 | 7.2381778 |
| ⋮ | ⋮ |
| 60 | 7.2360712 |
| ⋮ | ⋮ |
| 66 | 7.2360681 |

**Table 4 eigenvalues using Aitken method for the symmetric power method**

| Iteration m | Eigenvalue $\hat{u}^{(m)}$ |
|---|---|
| 1 | 2.1928251 |
| 2 | 6.9150522 |
| ⋮ | ⋮ |
| 10 | *5.2360813* |
| ⋮ | ⋮ |
| 20 | *5.2360541* |
| ⋮ | ⋮ |
| 30 | *5.2270442* |
| ⋮ | ⋮ |
| 40 | *7.2508583* |
| ⋮ | ⋮ |
| 50 | 7.2360907 |
| ⋮ | ⋮ |
| 60 | 7.2360681 |

**When the order of the matrix His 100x100**

**Table 5 eigenvalues using the power method**

| Iteration m | eigenvalue $u^{(m)}$ |
|---|---|
| 1 | 1.0000000 |
| 2 | 2.0000000 |
| ⋮ | ⋮ |
| 40 | 7.0623488 |
| ⋮ | ⋮ |
| 80 | 7.0641763 |
| ⋮ | ⋮ |
| 120 | 7.0642066 |
| ⋮ | ⋮ |
| 160 | 7.0654026 |
| ⋮ | ⋮ |
| 200 | 7.1127712 |
| ⋮ | ⋮ |
| 240 | 7.7478043 |
| ⋮ | ⋮ |
| 280 | 7.7585089 |
| ⋮ | ⋮ |
| 320 | 7.7587643 |
| ⋮ | ⋮ |
| 339 | 7.7587694 |

**Table 6 eigenvalues using the symmetric power method**

| Iteration m | Eigenvalue $u^{(m)}$ |
|---|---|
| 1 | 36.0000000 |
| 2 | 1.0000000 |
| ⋮ | ⋮ |
| 40 | 7.0641660 |
| ⋮ | ⋮ |
| 80 | 7.0643330 |
| ⋮ | ⋮ |
| 120 | 7.2644847 |
| ⋮ | ⋮ |
| 160 | 7.7578266 |
| ⋮ | ⋮ |
| 200 | 7.7587700 |

By applying the Aitken method to the approximated eigenvalues for acceleration the convergence to a large eigenvalue as follow:

**Table 7 eigenvalues using Aitken method for the power method**

| Iteration m | Eigenvalue $\tilde{u}^{(m)}$ |
|---|---|
| 1 | 0.0000000 |
| 2 | 3.7777778 |
| ⋮ | ⋮ |
| 40 | 7.0641809 |
| ⋮ | ⋮ |
| 80 | 7.0638345 |
| ⋮ | ⋮ |
| 120 | 8.1422186 |
| ⋮ | ⋮ |
| 160 | 7.7587725 |
| ⋮ | ⋮ |
| 167 | 7.7587710 |

**Table 8 eigenvalues using Aitken method for the symmetric power method**

| Iteration m | Eigenvalue $\tilde{u}^{(m)}$ |
|---|---|
| 1 | 2.1208151 |
| 2 | 7.5789474 |
| ⋮ | ⋮ |
| 40 | 7.0641732 |
| ⋮ | ⋮ |
| 80 | 7.0557540 |
| ⋮ | ⋮ |
| 120 | 7.7588549 |
| ⋮ | ⋮ |
| 147 | 7.7587710 |

After applying the power method, the symmetric power method and Aitken method for example 1 with 7 digitsaccuracy we have got the following results:

- When the matrix H of order 36x36 we found out the large eigenvalue at the step 126 of power method, in the accelerating approximation by Aitken method the large eigenvalue at the step 66. While the symmetric power method we got the approximate eigenvalue at the step 73, accelerating approximation by Aitken method the large eigenvalue at the step 60.
- When H of order 100x100 we have got the large eigenvalue at the step 339 of power method, , in the accelerating approximation by Aitken method the large eigenvalue at the step 167, While the symmetric power method we got the approximate eigenvalue at the step 200, in the accelerating approximation by Aitken method the large eigenvalue at the step 147.

The results of numerical tests of the power method and symmetric power method for example 2 are given in the following tables:

(i) When the order of the matrix H is 36x36

**Table 9 eigenvalues using the power method**

| Iteration m | Eigenvalue $u^{(m)}$ |
|---|---|
| 1 | 1.0000000 |
| 2 | 2.0000000 |
| ⋮ | ⋮ |
| 10 | 5.0823558 |
| ⋮ | ⋮ |
| 20 | 5.2354392 |
| ⋮ | ⋮ |
| 30 | 5.2360656 |
| ⋮ | ⋮ |
| 40 | 5.2360696 |
| ⋮ | ⋮ |
| 50 | 5.2361092 |
| ⋮ | ⋮ |
| 60 | 5.2371140 |
| ⋮ | ⋮ |
| 70 | 5.2623107 |
| ⋮ | ⋮ |
| 80 | 6.2787105 |
| ⋮ | ⋮ |
| 90 | 7.2026410 |
| ⋮ | ⋮ |
| 100 | 7.2347309 |
| ⋮ | ⋮ |
| 110 | 7.2360153 |
| ⋮ | ⋮ |
| 120 | 7.2360659 |
| ⋮ | ⋮ |
| 126 | 7.2360677 |

**Table 10 eigenvalues using the symmetric power method**

| Iteration m | Eigenvalue $u^{(m)}$ |
|---|---|
| 1 | 20.0000000 |
| 2 | 1.0000000 |
| ⋮ | ⋮ |
| 10 | 5.2202585 |
| ⋮ | ⋮ |
| 20 | 5.2360677 |
| ⋮ | ⋮ |
| 30 | 5.2361182 |
| ⋮ | ⋮ |
| 40 | 5.2679575 |
| ⋮ | ⋮ |
| 50 | 7.0615529 |
| ⋮ | ⋮ |
| 60 | 7.2357719 |
| ⋮ | ⋮ |
| 70 | 7.2360675 |
| ⋮ | ⋮ |
| 72 | 7.2360679 |

In the following step we will apply the Aitken method to the approximated eigenvalues for acceleration the convergence to a large eigenvalue

**Table 11 eigenvalues using Aitken method for the power method**

| Iteration m | Eigenvalue $\hat{u}^{(m)}$ |
|---|---|
| 1 | 0.0000000 |
| 2 | 3.7777778 |
| ⋮ | ⋮ |
| 10 | 5.2400189 |
| ⋮ | ⋮ |
| 20 | 5.2360680 |
| ⋮ | ⋮ |
| 30 | 5.2359710 |
| ⋮ | ⋮ |
| 40 | 5.1714694 |
| ⋮ | ⋮ |
| 50 | 7.2381778 |
| ⋮ | ⋮ |
| 60 | 7.2360712 |
| ⋮ | ⋮ |
| 65 | 7.2360681 |

**Table 12 eigenvalues using Aitken method for the symmetric power method**

| Iteration m | Eigenvalue $\tilde{u}^{(m)}$ |
|---|---|
| 1 | *2.1928251* |
| 2 | *6.9150522* |
| ⋮ | ⋮ |
| 10 | *5.2360813* |
| ⋮ | ⋮ |
| 20 | *5.2360541* |
| ⋮ | ⋮ |
| 30 | *5.2270442* |
| ⋮ | ⋮ |
| 40 | *7.2508583* |
| ⋮ | ⋮ |
| 50 | *7.2360907* |
| ⋮ | ⋮ |
| 58 | *7.2360681* |

(ii) When the order of the matrix H is 100x100

**Table 13 eigenvalues using the power method**

| Iteration m | eigenvalue $u^{(m)}$ |
|---|---|
| 1 | 1.0000000 |
| 2 | 2.0000000 |
| ⋮ | ⋮ |
| 40 | 7.0623488 |
| ⋮ | ⋮ |
| 80 | 7.0641766 |
| ⋮ | ⋮ |
| 120 | 7.0642094 |
| ⋮ | ⋮ |
| 160 | 7.0655227 |
| ⋮ | ⋮ |
| 200 | 7.1171846 |
| ⋮ | ⋮ |
| 240 | 7.7487719 |
| ⋮ | ⋮ |
| 280 | 7.7585323 |
| ⋮ | ⋮ |
| 320 | 7.7587649 |
| ⋮ | ⋮ |
| 339 | 7.7587694 |

**Table 14 eigenvalues using the symmetric power method**

| Iteration m | eigenvalue $u^{(m)}$ |
|---|---|
| 1 | 36.0000000 |
| 2 | 1.0000000 |
| ⋮ | ⋮ |
| 40 | 7.0641660 |
| ⋮ | ⋮ |
| 80 | 7.0643330 |
| ⋮ | ⋮ |
| 120 | 7.2644847 |
| ⋮ | ⋮ |
| 160 | 7.7578266 |
| ⋮ | ⋮ |
| 200 | 7.7587700 |

By applying the Aitken method to the approximated eigenvalues for acceleration the convergence to a large eigenvalue as follow:

**Table 15 eigenvalues using Aitken method for the power method**

| Iteration m | Eigenvalue $\tilde{u}^{(m)}$ |
|---|---|
| 1 | 0.0000000 |
| 2 | 3.7777778 |
| ⋮ | ⋮ |
| 40 | 7.0641809 |
| ⋮ | ⋮ |
| 80 | 7.0638345 |
| ⋮ | ⋮ |
| 120 | 7.9352722 |
| ⋮ | ⋮ |
| 160 | 7.7587716 |
| ⋮ | ⋮ |
| 164 | 7.7587710 |

**Table 16 eigenvalues using Aitken method for the symmetric power method**

| Iteration m | Eigenvalue $\tilde{u}^{(m)}$ |
|---|---|
| 1 | 2.1208151 |
| 2 | 7.5789474 |
| ⋮ | ⋮ |
| 40 | 7.0641732 |
| ⋮ | ⋮ |
| 80 | 7.0557540 |
| ⋮ | ⋮ |
| 120 | 7.7588549 |
| ⋮ | ⋮ |
| 147 | 7.7587710 |

After applying the power method, the symmetric power method and Aitken method for example 2 with 8 digits we have got the following results:

- When the matrix H of order 36x36 we found out the large eigenvalue at the step 126 of power method, in the accelerating approximation by Aitken method the large eigenvalue at the step 65. While the symmetric power method we got the approximate eigenvalue at the step 72, accelerating approximation by Aitken method the large eigenvalue at the step 58.
- When the matrix H of order 100x100 we have got the large eigenvalue at the step 339 of power method, , in the accelerating approximation by Aitken method the large eigenvalue at the step 164, While the symmetric power method we got the approximate eigenvalue at the step 200, in the accelerating approximation by Aitken method the large eigenvalue at the step 147.

**Conclusion.**

In the paper we studied power method, symmetric power method and Aitken method, applying them to the matrix H to find the large eigenvalue. The results reviewed that when the order of the matrix H increase the number of iterations increase. So the convergent to a large eigenvalue by the symmetric power methodis faster than the power method. While the convergent accelerate by using the Aitken methodyields to a better results.

**References**


[1].Aleyan O. A.: Applications of Finite Element Method to Solve Elliptic Partial Differential Equations Using Iterative Methods. An Academic, Intellectual, Cultural, Comprehensive and Arbitrated Journal Issued by the Faculty of Arts and Sciences in Zliten, Al-Mergeb. Issue No. 21 2012 AS –SAYIF (December) .

[2].Aleyan O. A.: On matrix splitting and application into iterative methods for linear systems An Academic, Intellectual, Cultural, Comprehensive and Arbitrated Journal Issued by the Faculty of Arts and Sciences in Zliten, Al-Merqab. Issue No. 18 2009 AS –SAYIF (JUNE) 1377.

[3].Berman, A. and Plemmons, R. Nonnegative matrices in the mathematical sciences, Academic Press, New York, 1979.

[4].Horn, R.A. and Johnson, C.R. Matrix analysis, Cambridge University Press, Cambridge, 1986.

[5].Ortega, J.M. and Rheinboldt, W.C. Iterative solution of linear systems, Plenum Press, New York, 1988.

[6].Miroslav Fiedler., Special matrices and their applications in numerical mathematics, Kluwer, 1986.

[7].Miroslav Fiedler and VlastimilPtak, on matrices with non-positive off-diagonal elements and positive principal minors, Czechosl. Math. Journal (87)12, 1962, pp 382400.

[8].Varga, R.S. Matrix iterative analysis, Prentice Hall, New York, 1962.

[9].Young, D. Iterative solution of large linear systems, Academic Press, New York, 1977.

[10]. Richard L. Burden and J. DonglasFaires. Numerical analysis, third edition prindle, Weber & Schmidt, Boston, 1985.