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THE EFFECT OF CORRELATION FUNCTIONS ON A NONLINEAR INTERACTION BETWEEN MULTISPECIES

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Abstract:-
This study provides the effect of correlation functions on the solution of nonlinear differential equations system (DES). For illustration, an application of the case of interacting biological system is introduced for multispecies population, Moreover, the concept of stability and identification of equilibrium points are studied, this obtain the analytic approximate solutions. In order to evaluate the solution of the correlation functions, this study has been limited on the chronological correlation functions of the two type's; auto and cross. That is to reach the expense of second and third-correlation functions.

Keywords:-*Nonlinear (DES), Interacting biological system, Multispecies population, Analytic approximate solutions, Auto and cross correlation functions.*

INTRODUCTION

The Lotka-Volterra model [1, 2] is widely used to study the dynamics of interacting species in ecology and elsewhere [3]. Kapur [4] says in his literature, we are not first restricted to the use of mathematical techniques already known. In fact, one role of mathematicians interested in biological and medical problems is to evolve new mathematical methods for dealing with the complex situations in life sciences [5]. In order to, analyzed the stability of continuous and discrete population models, he examined the stability of models for interacting species with same points of equilibrium, using difference equation system. [6] Studied moments for some general birth death processes, he determined the moments of all orders for the probability distribution of ultimate size of the population the generalized birth and death processes with twin births being studied in [7] where he used the difference equation technique. It had been shown in [8] that equilibrium point for predator-prey models with discrete time lage is always unstable. the use of hypergeometric functions to generalized Birth and death processes is given by[9] he enumerates the number of possible relations between probabilities of ultimate extinction of birth and death processes. In his paper titled some mathematical models for population growth, he decided one of the most successful models for explaining the growth of populations of bacteria and even of humans is the so-called Logistic model [10].

The combined use of game theory and modified volterra equations in describing the population dynamics is treated by [11] that concluded that volterr's system of differential equations for n interacting species has been modified and it is shown that the modified system is equivalent to the system of cubic differential equations obtained earlier for animal conflicts from considerations of theory of games. In his study of nonlinear continuous- time discrete- age- scale population models, he showed that whenever the corresponding linear model predicts exponential growth, the nonlinear model gives a stable equilibrium age-distribution [12]. The study of the effect of Harvesting on competing population is due to [13], it showed that for the simplest competition model for two species there are four non-degenerate possibilities for ultimate behavior according as (i) the first species alone survives, (ii) the second species alone survives, (iii) the two species coexist, in stable equilibrium and (iv) the two species coexist in unstable equilibrium, and the survival of first or second species depends on the initial population size. Another study [14] is treated the optimal foraging and predator-prey dynamics. And [15] treated the periodicity in a delayed ratio-dependent predator-prey system. Lin QIV and Taketono MITSUI [16] discussed the problem of predator-prey dynamics with delay when prey dispersing in n-patch environment. Previous study has proposed a model to describe the interaction between a diseased fish population and their predators he analyzed the stability of equilibrium points for a large range of parameter values [17]. Paul [18] develops a mathematical model of a biological arms race between a class of predators and a class of prey where the prey is dangerous to the predators, Ross Cresman [19] studied the evolutionary stability in LotkaVolterra system. And [20] is studied the global stability of a predator-prey system with stage stricture for the predators. Previously manipulated the spatial dynamics and cross-correlation in a transient predator-prey system [21]. The effects of correlated interactions in a biological convolution model with individual-based dynamics is a research given by [22]. Rozenleld [23] introduces the concept of coherence into the predation of biological system through his work titled on the influence of noise on the critical and oscillatory behavior of a predator-prey Models: coherent stochastic resonance at the proper frequency of the system. Both [24, 25] introduces vibrational iteration method for solving multispecies lotka-volterra equations. Nicola [26] has proposed utility functions and lotka-volterra model: A possible connection in predator-prey game. Susmitapaul [27] introduces numerical solution of lotka-volterra prey predator model by using runge-kutta Fehlberg method and Laplace a domain decomposition method. In addition [28] discuss application of perturbation-iteration method to lotka-volterra equations. Nourataher in [29] studied analysis of hybrid dynamical systems with an application in biological systems.

In this paper we shall study the two types of interaction between more than two species of Biological system. Namely, the predation and the compition via Lotkavolterra, besides we represent a mixed model for the predation and competition. Also we shall generalize the two species case to investigate the effects of interactions among any number of species. Moreover, the concept of stability and identification of equilibrium points are studied, this obtain the analytic approximate solutions. In order to evaluate the solution of the correlation functions.

THE BASIC SYSTEM OF EQUATIONS OF VOLTERRA'S MODEL

The basic system of equations for ninteracting species can be written as

$$\frac{dN_i}{dt} = k_i N_i + \sum_{j=1}^{n} b_{ij} N_i N_j \quad (i = 1,2,\ldots,n). \qquad \textbf{(2.1)}$$

Due to interaction between the i-th and j-th species, the changes in the two populations per unit time are given by $b_{ij}N_iN_j$ and $b_{ji}N_iN_j$, respectively. Assuming that all interactions are of the predator–prey type b_{ij} and b_{ji} have opposite sings. Moreover, if the ratio of changes in the two populations is c_i/c_j, then we can write,

$$b_{ij} = c_i a_{ij},\ b_{ji} = c_j a_{ji},\ a_{ij} = -a_{ji}\ (i \neq j) \qquad \textbf{(2.2)}$$

so that (2.1)becomes

$$\frac{dN_i}{dt} = k_i N_i + c_i N_i \sum_{j=1}^{n} a_{ij} N_j = k_i N_i + \beta_i^{-1} N_i \sum_{j=1}^{n} a_{ij} N_j\ ,\ (i = 1,2,\ldots,n) \qquad \textbf{(2.3)}$$

We also assume that the Verhulst term or the resource-limited term is absent for each species so that aii= 0for all i. Using(2.2), we find that the matrix(aij)is skew symmetric.Let qidenote the equilibrium or the steady-state value of Ni. Then (2.3) gives

$$q_i k_i + \beta_i^{-1} q_i \sum_{j=1}^{n} a_{ij} q_j = 0 \quad (i = 1,2,\ldots,n). \qquad \textbf{(2.4)}$$

If none of the q'sis zero, we get

$$k_i \beta_i + \sum_{j=1}^{n} a_{ij} q_j = 0 \; (i = 1,2,\ldots,n). \qquad \textbf{(2.5)}$$

Equation (2.5) gives unique finite values for q1, q2, ...,qnif the determinant of the matrix

$$a_{ij}) \text{ is } |A| = |a_{ij}| \neq 0. \qquad \textbf{(2.6)}$$

However, we know that the determinant of a skew-symmetric matrix of odd order is always zero so that (2.5) does not give unique finite values for q1, q2, ...,qnif nis odd. Our model will therefore have non-zero equilibrium population sizes only when the number of species is even and $|A| \neq 0$.
We shall also assume that the parameters have such values that qi's come out to be positive.
Again, multiplying (2.5) by qi and summing for all i, we get

$$\sum_{i=1}^{n} k_i \beta_i q_i = -\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} q_i q_j = 0 \qquad \textbf{(2.7)}$$

Since the right-hand side of (2.7) vanishes because of the skew-symmetry of (aij). Since, $\beta_i > 0,$ and assuming all qi's to be positive, all ki's cannot be positive

Existence of Constant of Motion

$$\text{Let}(i = 1,2,\ldots,n) \quad \upsilon_i = \ln \frac{N_i}{q_i} \qquad \textbf{(3.1)}$$

Where logarithm here and throughout this chapter is to the base e. Then we easily see that

$$\upsilon_i \geq 0 \text{ or } \leq 0 \Leftrightarrow N_i \geq q_i \text{ or} \leq q_i \qquad \textbf{(3.2)}$$

so that υ_i may be regarded as a measure of the departure of Ni from qi.
Substituting from (3.1) in (2.3), we get

$$\beta_i \frac{d\upsilon_i}{dt} = k_i \beta_i + \sum_{j=1}^{n} a_{ij} q_j \exp \upsilon_j \; (i = 1,2,\ldots,n). \qquad \textbf{(3.3)}$$

From (2.5) and (3.3):

$$\beta_i \frac{d\upsilon_i}{dt} = \sum_{j=1}^{n} a_{ij} q_j (\exp \upsilon_j - 1), \; (i = 1,2,\ldots,n). \qquad \textbf{(3.4)}$$

Multiplying both sides by $q_i(\exp \upsilon_i - 1)$ and summing for i from 1 ton, we get

$$\sum_{i=1}^{n} \beta_i q_i (\exp \upsilon_i - 1) \frac{d\upsilon_i}{dt} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} q_i q_j (\exp \upsilon_i - 1)(\exp \upsilon_j - 1). \quad \textbf{(3.5)}$$

On account of the skew-symmetry of the matrix (aij), the right-hand side of (3.5) vanishes

$$\text{so that} \sum_{i=1}^{n} \beta_i q_i (\exp \upsilon_i - 1) \frac{d\upsilon_i}{dt} = 0. \qquad \textbf{(3.6)}$$

Integrating (3.6), we obtain

$$G = \sum_{i=1}^{n} \beta_i q_i (\exp \upsilon_i - \upsilon_i) = \sum_{i=1}^{n} G_i = constant = \sum_{i=1}^{n} \beta_i q_i \left(\frac{N_{i0}}{q_i} - \ln \frac{N_{i0}}{q_i}\right), \qquad \textbf{(3.7)}$$

Where Ni0 is the initial value of Ni .Thus there exists a constant of motion which is the sum of a number of similar components. Each of these components is positive since, whe $\upsilon_i \geq 0, \exp \upsilon_i > \upsilon_i,$ and When $\upsilon_i < 0,$ both $\exp \upsilon_i$ and $\exp(-\upsilon_i)$ Are positive. As time changes, each component Gi changes, but the sum of Gi remains constant and finite. Gi increases or decreases according as the population of the i-th species increases or decreases since $dG_i / d\upsilon_i > 0.$ As time progresses, populations of some species increase, while populations of other species decrease in such a way that G does not change. G does of course depend on the initial populations of the n species. Also, since G is finite, the population of every species remains finite.
From (2.3), (2.5), and (3.7)

$$\frac{dG}{dt} = \frac{d}{dt}\sum_{i=1}^{n}\beta_i(N_i - q_i \ln\frac{N_i}{q_i}) = \sum_{i=1}^{n}\beta_i(1-\frac{q_i}{N_i})\frac{dN_i}{dt}$$

$$= \sum_{i=1}^{n}\beta_i k_i(N_i - q_i) + \sum_{i=1}^{n}(1-\frac{q_i}{N_i})\sum_{j=1}^{n}a_{ij}N_iN_j$$

$$= -\sum_{i=1}^{n}(N_i - q_i)\sum_{j=1}^{n}a_{ij}q_j + \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}N_iN_j - \sum_{j=1}^{n}\sum_{i=1}^{n}a_{ij}q_iN_j$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}+a_{ji})N_iq_j + \sum_{j=1}^{n}\sum_{i=1}^{n}a_{ij}q_iq_j + \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}N_iN_j. \quad \textbf{(3.8)}$$

All the terms on the right-hand side of (3.8) vanish because of the skew-symmetry of (aij), and this again verifies that G is a constant of motion. Further, if $a_{ij} = -a_{ji}$ when $j \neq i$ but $a_{ii} \neq 0$ for all i, we get

$$\frac{dG}{dt} = -2\sum_{i=1}^{n}a_{ii}N_iq_i + \sum_{i=1}^{n}a_{ii}(N_i^2 + q_i^2) = \sum_{i=1}^{n}a_{ii}(N_i - q_i)^2 \qquad \textbf{(3.9)}$$

Since the resource-limiting coefficient aii always less than or equal to zero, we find that

$$dG/dt \leq 0 \qquad \textbf{(3.10)}$$

So that, in the case when $a_{ii} \neq 0$; for all i, G is not a constant but a monotonic decreasing function of t. The maximum value of G occurs when Ni=Ni0 and the minimum value occurs when Ni=qi for all i. Stability of Equilibrium Point We know that, in the equilibrium point $N_i = q_i, v_i = 0$, and so from (3.4)

$$\beta_i\frac{d\upsilon_i}{dt} = \sum_{j=1}^{n}a_{ij}q_j\upsilon_j \;\; (i = 1,2,...,n) \qquad \textbf{(4.1)}$$

Or $$\frac{\beta_i}{q_i}\frac{dv_i}{dt} = \sum_{j=1}^{n}a_{ij}v_j \;\; (v_j = q_j\upsilon_j, q_j \neq 0). \qquad \textbf{(4.2)}$$

The community matrix, whose eigenvalues determine the stability, is

$$\begin{bmatrix} 0 & a_{12}\frac{q_1}{\beta_1} & a_{13}\frac{q_1}{\beta_1}\cdots & a_{1n}\frac{q_1}{\beta_1} \\ a_{21}\frac{q_2}{\beta_2} & 0 & a_{23}\frac{q_2}{\beta_2}\cdots & a_{2n}\frac{q_2}{\beta_2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}\frac{q_n}{\beta_n} & a_{n2}\frac{q_n}{\beta_n} & a_{n3}\frac{q_n}{\beta_n}\cdots & 0 \end{bmatrix} = \begin{bmatrix} \frac{q_1}{\beta_1} & 0 & 0\ldots & 0 \\ 0 & \frac{q_2}{\beta_2} & 0\ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0\ldots & \frac{q_n}{\beta_n} \end{bmatrix} \begin{bmatrix} 0 & a_{12} & a_{13}\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23}\cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3}\cdots & a_{nn} \end{bmatrix} = DA. \quad \textbf{(4.3)}$$

This matrix is the product of a symmetric matrix D and a skew-symmetric matrix A and, therefore, all its eigenvalues are purely imaginary or zero. However, since we are taking n to be even and $|A| \neq 0$, all the eigenvalues in our matrix are purely imaginary so that the secular equation is of the from

$$\textbf{(4.4)} \qquad (\lambda^2 + \omega_1^2)(\lambda^2 + \omega_2^2)\ldots(\lambda^2 + \omega_{n/2}^2) = 0$$

which gives $$\upsilon_i = \sum_{j=1}^{n/2}(A_{ij}\cos(\omega_j t) + \beta_{ij}\sin(\omega_j t)). \qquad \textbf{(4.5)}$$

Now from (4..2) $$\sum_{i=1}^{n}\frac{\beta_i}{q_1}v_i\frac{dv_i}{dt} = \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}v_iv_j = 0 \qquad \textbf{(4.6)}$$

Because of the skew-symmetry of (aij). Integrating (4.6), we get

$$\sum_{i=1}^{n}\frac{\beta_i}{q_i}v_i^2 = \sum_{i=1}^{n}\beta_iq_i\upsilon_i^2 = cons\tan t \qquad \textbf{(4.7)}$$

Which is a hyper ellipsoid in the n-dimensional space with v1, v2,...v_n as the coordinates. If we write $N_i = q_i + u_i$, then (4.7) gives, on retaining the lowest powers ui

$$\sum_{i=1}^{n}\beta_iq_i(\ln\frac{N_i}{q_i})^2 = \sum_{i=1}^{n}\beta_iq_i(\ln(1+\frac{u_i}{q_i}))^2 = \sum_{i=1}^{n}\frac{\beta_i}{q_i}u_i^2 = \sum_{i=1}^{n}\frac{\beta_i}{q_i}(N_i - q_i)^2 = cons\tan t \quad \textbf{(4.8)}$$

x which is again a hyper ellipsoid with center q1, q2,...,qnin n-dimensional space. Thus, for small perturbations from the equilibrium point, the trajectory always lies on a closed hypersurface. This shows that the equilibrium is neutral and conservative oscillations occur about the equilibrium point. Even when the perturbations are not small, or when (3.7) is used, the trajectory lies on the hypersurface.

$$\sum_{i=1}^{n} \beta_i q_i \left(\frac{N_i}{q_i} - \ln \frac{N_i}{q_i}\right) = \sum_{i=1}^{n} \beta_i q_i \left(\frac{N_{i0}}{q_i} - \ln \frac{N_{i0}}{q_i}\right) \quad (4.9)$$

Which is a closed hypersurface meeting every straight line parallel to the axes in two points. The trajectory need not be closed except for n = 2.

Even for small oscillations, the periods are $2\pi/\omega_1, 2\pi/\omega_2, \ldots, 2\pi/\omega_{n/2}$, and there may not be any period which is an integral multiple of all these periods. Thus, for a four species model, if the periods are $2\pi/3$ and $2\pi/5$, then, at time $t = 15$, the trajectory for small oscillations will return to the initial point, but if the periods are $2\pi/\sqrt{2}$ and $2\pi/\sqrt{3}$, then the trajectory will never return to the starting point though it may come arbitrarily close to it. Similar results hold when oscillations are not small. Thus, in general, the trajectory always lies on a closed hypersurface, but it is not closed except when n = 2. The trajectory point goes on moving round and round the hypersurface (4.9) without its ever returning to the original point. Moreover, the motion is also not strictly periodic except when n=2.

The foregoing discussion shows that the result about the neutrality of equilibrium point holds not only for local or neighbourhood stability, but also for global stability. This is confirmed by the existence of a function G which is always positive and whose derivative G/dt is negative semi-definite.. Thus, G acts as a Lypunov function. When aii= 0,it is seen that there are conservative oscillations about the equilibrium point irrespective of where the initial points are; when $a_{ii} \neq 0$, the matrix Ahas at least some non-zero diagonal elements and is not skew-symmetric. The eigenvalues of the community matrix (4.3)need not be purely imaginary, and the equilibrium is no longer neutral. As already noted, when n=2, the equilibrium is locally stable and, because of the existence of the Lypunov function G, it is globally stable. Thus the non-zero values of aii, however small they may be, change the nature of the equilibrium. This is expressed by saying that volterra's model is structurally unstable. A model is said to be structurally stable if small changes in its parameters do not change the nature of the equilibrium point. Volterra's model is structurally stable if we consider small changes in aij's $(i \neq j)$ only. but it is structurally unstable if we introduce Verhulst terms (however small these may be) because, without these terms, there is neutral equilibrium ; with these terms, the equilibrium may be stable or unstable. However, we may note that, whether aij is zero or not, the results of local stability analysis extend to global stability analysis because of the existence of the Lypunov function.

Solutions of Volterra's Model for n Interacting Species

The basic system of equations for n interacting species can be written as

$$\frac{dN_i}{dt} = k_i N_i + \sum_{j=1}^{n} b_{ij} N_i N_j \quad (i = 1,2,\ldots,n) \quad (5.1)$$

at i , j = 1,2,3 and $i \neq j$ the model can be written as

$$\frac{dN_i}{dt} = k_i N_i + \sum_{j=1, i \neq j}^{3} b_{ij} N_i N_j \quad (i = 1,2,3) \quad (5.2)$$

$$\frac{dN_1}{dt} = K_1 N_1 + b_{12} N_1 N_2 + b_{13} N_1 N_3,$$

Then,

$$\frac{dN_2}{dt} = K_2 N_2 + b_{21} N_1 N_2 + b_{23} N_2 N_3, \quad (5.3)$$

$$\frac{dN_3}{dt} = K_3 N_3 + b_{31} N_1 N_3 + b_{32} N_2 N_3.$$

Let $N_1(t) = N_{10} + \varepsilon_1 \eta_1(t)$, $N_2(t) = N_{20} + \varepsilon_2 \eta_2(t)$ and $N_3(t) = N_{30} + \varepsilon_3 \eta_3(t)$ (5.4)

Then and neglecting cross terms $\eta_i \eta_J$, $i \neq J = 1,2,3$ we have

$$\frac{d\eta_1}{dt} = \alpha_1 \eta_1(t) + \beta_1 \eta_2(t) + \gamma_1 \eta_3(t) + \omega_1 \quad (5.5)$$

where: $\alpha_1 = K_1 + b_{12} N_{20} + b_{13} N_{30}$, $\beta_1 = \frac{b_{12} N_{10} \varepsilon_2}{\varepsilon_1}$, $\gamma_1 = \frac{b_{13} N_{10} \varepsilon_3}{\varepsilon_1}$ and

$$\omega_1 = \frac{K_1 N_{10} + b_{12} N_{10} N_{20} + b_{13} N_1 N_3}{\varepsilon_1}, \quad \frac{d\eta_2}{dt} = \alpha_2 \eta_1(t) + \beta_2 \eta_2(t) + \gamma_2 \eta_3(t) + \omega_2 \quad (5.6)$$

where: $\alpha_2 = \frac{b_{21} N_{20} \varepsilon_1}{\varepsilon_2}$, $\beta_2 = K_2 + b_{21} N_{10} + b_{23} N_{30}$, $\gamma_2 = \frac{b_{23} N_{20} \varepsilon_3}{\varepsilon_2}$ and

$$\omega_2 = \frac{K_2 N_{20} + b_{21} N_{10} N_{20} + b_{23} N_{20} N_{30}}{\varepsilon_2}$$

$$\frac{d\eta_3}{dt} = \alpha_3 \eta_1(t) + \beta_3 \eta_2(t) + \gamma_3 \eta_3(t) + \omega_3 \quad (5.7)$$

Where: $\alpha_3 = \frac{b_{31} N_{30} \varepsilon_1}{\varepsilon_3}$, $\beta_3 = \frac{b_{32} N_{30} \varepsilon_2}{\varepsilon_3}$, $\gamma_3 = K_3 + b_{31} N_{10} + b_{32} N_{20}$ and

$$\omega_3 = \frac{K_3 N_{30} + b_{31} N_{10} N_{30} + b_{32} N_{20} N_{30}}{\varepsilon_3}$$

In matrix form equations (5.5) , (5.6)and (5.7) take the form

$$\frac{d}{dt}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}=\begin{pmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}+\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} \qquad (5.8)$$

We shall solve the equation

$$\frac{d}{dt}\begin{pmatrix}\eta_1+\Omega_1\\ \eta_2+\Omega_2\\ \eta_3+\Omega_3\end{pmatrix}=A\begin{pmatrix}\eta_1+\Omega_1\\ \eta_2+\Omega_2\\ \eta_3+\Omega_3\end{pmatrix} \qquad (5.9)$$

where: $A=\begin{pmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{pmatrix}$ and $\begin{pmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{pmatrix}=A^{-1}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}$

Assuming that $$(\eta+\Omega)(t)=\varepsilon e^{\lambda t} \qquad (5.9a)$$

Leads to the set of linear algebraic equations

$$\begin{pmatrix}\alpha_1-\lambda & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2-\lambda & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3-\lambda\end{pmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{12}\\ \varepsilon_{13}\end{pmatrix}=0 \qquad (5.9b)$$

Which determine the eigenvalues and eigenvectors of A.
Now the eigenvalues of Aare given by

$$|A-\lambda I|=0 \qquad (5.9c)$$

Or $$\begin{vmatrix}\alpha_1-\lambda & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2-\lambda & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3-\lambda\end{vmatrix}=0 \qquad (5.9d)$$

(i.e) $$\lambda^3-a\lambda^2-b\lambda-c=0 \qquad (5.9e)$$

Where: $a=\alpha_1-\beta_2+\gamma_3$, $b=\beta_3\gamma_2-\beta_2\gamma_3-\alpha_1\beta_2-\alpha_1\gamma_3+\alpha_2\beta_1+\gamma_1\alpha_3$ and

$$c=\alpha_1\beta_2\gamma_3-\alpha_1\beta_3\gamma_2+\alpha_3\beta_1\gamma_2-\alpha_2\beta_1\gamma_3+\alpha_2\beta_3\gamma_1-\alpha_3\beta_2\gamma_1.$$

The roots of cubic equation are

$$\lambda_1=A_3,\ \lambda_2=\lambda_{21}+i\lambda_{22} \text{ and } \lambda_3=\lambda_{21}-i\lambda_{22}$$

where: $A_3=\sqrt[3]{A_1}+\sqrt[3]{A_2}$, $\lambda_{21}=-\dfrac{\sqrt[3]{A_1}+\sqrt[3]{A_2}}{2}$ and $\lambda_{22}=\sqrt{3}\,\dfrac{\sqrt[3]{A_1}-\sqrt[3]{A_2}}{2}$,

Since: $A_1=\dfrac{-q}{2}+\sqrt{\dfrac{q^2}{4}+\dfrac{p^3}{27}}$, $A_2=\dfrac{-q}{2}-\sqrt{\dfrac{q^2}{4}+\dfrac{p^3}{27}}$, $p=-b+\dfrac{a^2}{3}$

$$q=-c-\frac{ba}{3}-\frac{2a^3}{27}.$$

Then the eigenvector are

$$\varepsilon^{(1)}=\begin{pmatrix}a_{11}\\ 1\\ 0\end{pmatrix},\ \varepsilon^{(2)}=\begin{pmatrix}a_{12}-ib_{12}\\ 1\\ 0\end{pmatrix} \text{ and } \varepsilon^{(3)}=\begin{pmatrix}a_{12}+ib_{12}\\ 1\\ 0\end{pmatrix} \qquad (5.10)$$

where $a_{11}=\dfrac{-\beta_1}{\alpha_1-A_3}$, $a_{12}=\dfrac{-\beta_1(\alpha_1-\lambda_{21})}{(\alpha_1-\lambda_{21})^2+\lambda_{22}^2}$ and $b_{12}=\dfrac{\beta_1\lambda_{22}}{(\alpha_1-\lambda_{21})^2+\lambda_{22}^2}$

From (5.10) into (5.9a) we get: $$(\eta+\Omega)^{(1)}(t)=\begin{pmatrix}a_{11}\\ 1\\ 0\end{pmatrix}e^{A_3 t}, \qquad (5.10a)$$

$$(\eta+\Omega)^{(2)}(t)=\begin{pmatrix}a_{12}-ib_{12}\\ 1\\ 0\end{pmatrix}e^{(\lambda_{21}+i\lambda_{22})t}, \qquad (5.10b)$$

$$(\eta+\Omega)^{(3)}(t)=\begin{pmatrix}a_{12}+ib_{12}\\ 1\\ 0\end{pmatrix}e^{(\lambda_{21}-i\lambda_{22})t}. \qquad (5.10c)$$

The general solution of the equation (5.9) are

$$(\eta+\Omega)(t)=c_1\begin{pmatrix}a_{11}\\1\\0\end{pmatrix}e^{A_1t}+c_2\begin{pmatrix}a_{12}\cos\lambda_{22}t+b_{12}\sin\lambda_{22}t\\\cos\lambda_{22}t\\0\end{pmatrix}e^{\lambda_{21}t}+$$

$$c_3\begin{pmatrix}-b_{12}\cos\lambda_{22}t+a_{12}\sin\lambda_{22}t\\\sin\lambda_{22}t\\0\end{pmatrix}e^{\lambda_{21}t} \quad (5.11)$$

Then $\eta_1(t)=C_1e^{A_1t}+C_2\cos\lambda_{22}te^{\lambda_{21}t}+C_3\sin\lambda_{22}te^{\lambda_{21}t}-\Omega_1,$

$$\eta_2(t)=c_1e^{A_1t}+c_2\cos\lambda_{22}te^{\lambda_{12}t}+c_3\sin\lambda_{22}te^{\lambda_{12}t}-\Omega_2, \quad (5.12)$$

$$\eta_3(t)=-\Omega_3$$

where: $C_1=c_1a_{11}$, $C_2=c_2a_{12}-c_3b_{12}$ and $C_3=c_2b_{12}+c_3a_{12}$.

Form equation (5.12) into (5.4) we get:

$$N_1(t)=d_1e^{A_1t}+d_2\cos\lambda_{22}te^{\lambda_{21}t}+d_3\sin\lambda_{22}te^{\lambda_{21}t}+d_4,$$

$$N_2(t)=D_1e^{A_1t}+D_2\cos\lambda_{22}te^{\lambda_{21}t}+D_3\sin\lambda_{22}te^{\lambda_{21}t}+D_4, \quad (5.13)$$

$$N_3(t)=D_1'.$$

Where: $d_1=\varepsilon_1C_1, d_2=\varepsilon_1C_2, d_3=\varepsilon_1C_3, d_4=N_{10}-\varepsilon_1\Omega_1,$

$$D_1=\varepsilon_2c_1, D_2=\varepsilon_2c_2, D_3=\varepsilon_2c_3, D_4=N_{20}-\varepsilon_2\Omega_2$$

and $D_1'=N_{30}-\varepsilon_3\Omega_3$.

We can treat the nonlinear problem iteratively considering N1 (t), N2 (t) and N3 (t) as a 1st approximation $(N_1^{(1)}(t),N_2^{(1)}(t),N_3^{(1)}(t))$ then the 2nditerative solution can be obtained from:

$$\frac{dN_1^{(2)}}{dt}=K_1N_1^{(1)}+b_{12}N_1^{(1)}N_2^{(1)}+b_{13}N_1^{(1)}N_3^{(1)},$$

$$\frac{dN_2^{(2)}}{dt}=K_2N_2^{(1)}+b_{21}N_1^{(1)}N_2^{(1)}+b_{23}N_2^{(1)}N_3^{(1)}, \quad (5.14)$$

$$\frac{dN_3^{(2)}}{dt}=K_3N_3^{(1)}+b_{31}N_1^{(1)}N_3^{(1)}+b_{32}N_2^{(1)}N_3^{(1)}.$$

From equation (5.13) in to (5.14) and integrating we obtain $N_1^{(2)}(t),N_2^{(2)}(t)$ and $N_3^{(2)}(t)$ similarly can be obtained the 3rd approxim $(N_1^{(3)}(t),N_2^{(3)}(t),N_3^{(3)}(t))$

Second and Third-Order Correlation Functions

Besides the pair correlation function $C_{ij}^{(2)}(t)$ which measures the interaction between pairs of species there is another fine interaction between triplets of species defined b

$$C_{ijk}^{(3)}(t)=\langle\Delta N_i(t)\Delta N_j(t)\Delta N_k(t)\rangle$$

$$=\langle(N_i(t)-N_{i0})(N_j(t)-N_{j0})(N_k(t)-N_{k0})\rangle$$

$$=\langle N_i(t)N_j(t)N_k(t)\rangle-\sum_{i\neq j\neq k=1,2,3}N_{i0}\langle N_j(t)N_k(t)\rangle+$$

$$\sum_{i\neq j\neq k=1,2,3}N_{i0}N_{j0}\langle N_k(t)\rangle-N_{i0}N_{j0}N_{k0}. \quad (6.1)$$

Where: $i\neq j\neq k=1,2,3$

In our study the expected value of a function f (t) is defined by

$$\langle f(t)\rangle=\frac{1}{T}\int_A^{A+T}f(t)dt \quad (6,2)$$

using equation (6,2) for $\langle f(t)\rangle$ and from equation (5.13) we have

$$\langle N_1(t)\rangle=k_1+\frac{1}{T}[k_2e^{A_1T}+k_3\cos\lambda_{22}Te^{\lambda_{21}T}+k_4\sin\lambda_{22}Te^{\lambda_{21}T}-k_5] \quad (6.3)$$

where: $k_1 = d_4$, $k_2 = \frac{d_1}{A_3}$, $k_3 = \frac{d_2\lambda_{21} - d_3\lambda_{22}}{\lambda_{11}^2 + \lambda_{22}^2}$, $k_4 = \frac{d_2\lambda_{22} + d_3\lambda_{21}}{\lambda_{11}^2 + \lambda_{22}^2}$,

and $k_5 = \frac{d_1}{A_3} + \frac{d_2\lambda_{22} - d_3\lambda_{21}}{\lambda_{11}^2 + \lambda_{22}^2}$.

$$\langle N_2(t)\rangle = L_1 + \frac{1}{T}[L_2 e^{A_3 T} + L_3 \cos\lambda_{22}Te^{\lambda_{21}T} + L_4 \sin\lambda_{22}Te^{\lambda_{21}T} - L_5] \quad \textbf{(6.4)}$$

where: $L_1 = D_4$, $L_2 = \frac{D_1}{A_3}$, $L_3 = \frac{D_2\lambda_{21} - D_3\lambda_{22}}{\lambda_{21}^2 + \lambda_{22}^2}$, $L_4 = \frac{D_2\lambda_{22} + D_3\lambda_{21}}{\lambda_{21}^2 + \lambda_{22}^2}$,

and $L_5 = \frac{D_1}{A_3} + \frac{D_2\lambda_{21} - D_3\lambda_{22}}{\lambda_{21}^2 + \lambda_{22}^2}$,

(6.5)

$$\langle N_3(t)\rangle = D_1',$$

$$\langle N_1(t)N_2(t)\rangle = H_1 + \frac{1}{T}[H_2 e^{A_3 T} + H_3 e^{2A_3 T} + H_4 e^{2\lambda_{21}T} + H_5 \cos\lambda_{22}Te^{\lambda_{21}T} +$$

$$H_6 \sin\lambda_{22}Te^{\lambda_{21}T} + H_7 \cos\lambda_{22}Te^{\lambda_{23}T} + H_8 \sin\lambda_{22}Te^{\lambda_{23}T} + H_9 \cos\lambda_{22}Te^{2\lambda_{23}T}$$

$$+ H_{10} \sin\lambda_{22}Te^{2\lambda_{23}T} - H_{11}] \quad \textbf{(6.6)}$$

where: $H_1 = G_1$, $H_2 = \frac{G_1}{A_3}$, $H_3 = \frac{G_3}{2A_3}$, $H_4 = \frac{G_4}{2\lambda_{21}}$,

$H_5 = \frac{G_5\lambda_{21} - G_6\lambda_{22}}{\lambda_{21}^2 + \lambda_{22}^2}$, $H_6 = \frac{G_5\lambda_{22} - G_6\lambda_{21}}{\lambda_{21}^2 + \lambda_{22}^2}$,

$H_7 = \frac{G_7\lambda_{23} - G_8\lambda_{22}}{\lambda_{23}^2 + \lambda_{22}^2}$, $H_8 = \frac{G_7\lambda_{22} + G_8\lambda_{22}}{\lambda_{23}^2 + \lambda_{22}^2}$, $H_9 = \frac{G_9\lambda_{21} - G_{10}\lambda_{22}}{2(\lambda_{21}^2 + \lambda_{22}^2)}$,

$H_{10} = \frac{G_9\lambda_{22} + G_{10}\lambda_{21}}{2(\lambda_{21}^2 + \lambda_{22}^2)}$ and $H_{11} = H_2 + H_3 + H_4 + H_5 + H_7 + H_9$.

since: $G_1 = d_4 D_4$, $G_2 = d_1 D_4 + d_4 D_1$, $G_3 = d_1 D_1$, $G_4 = \frac{d_2 D_2 + d_3 D_3}{2}$,

$G_5 = d_2 D_4 + d_4 D_2$, $G_6 = d_3 D_4 + d_4 D_2$, $G_7 = d_1 D_2 + d_2 D_1$,

$G_8 = d_1 D_3 + d_3 D_1$, $G_9 = \frac{d_2 D_2 - d_3 D_3}{2}$, $G_{10} = \frac{d_2 D_3 + d_3 D_2}{2}$ and

$\lambda_{23} = A_3 + \lambda_{21}$.

and $\langle N_1(t)N_3(t)\rangle = I_1 + \frac{1}{T}[I_2 e^{A_3 T} + I_3 \cos\lambda_{22}Te^{\lambda_{21}T} + I_4 \sin\lambda_{22}Te^{\lambda_{21}T} - I_5]$ **(6.7)**

Where: $I_1 = D_1' d_4$, $I_2 = \frac{D_1' d_1}{A_3}$, $I_3 = \left(\frac{d_2\lambda_{21} - d_3\lambda_{22}}{\lambda_{21}^2 + \lambda_{22}^2}\right)$,

$I_4 = D_1'\left(\frac{d_2\lambda_{22} + d_3\lambda_{21}}{\lambda_{21}^2 + \lambda_{22}^2}\right)$ and $I_5 = D_1'[\frac{d_1}{A_3} + \frac{d_2\lambda_{21} - d_3\lambda_{22}}{\lambda_{21}^2 + \lambda_{22}^2}]$

$$\langle N_2(t)N_3(t)\rangle = J_1 + \frac{1}{T}[J_2 e^{A_3 T} + J_3 \cos\lambda_{22}Te^{\lambda_{21}T} + J_4 \sin\lambda_{22}Te^{\lambda_{21}T} - J_5] \quad \textbf{(6.8)}$$

$$\text{Where: } J_1 = D_1' d_4,\ J_2 = \frac{D_1' d_1}{A_3},\ J_3 = D_1'\left(\frac{D_2\lambda_{21} - D_3\lambda_{22}}{\lambda_{21}^2 + \lambda_{22}^2}\right),$$

$$J_4 = D_1'\left(\frac{D_2\lambda_{22} + D_3\lambda_{21}}{\lambda_{21}^2 + \lambda_{22}^2}\right) \text{ and } J_5 = D_1'\left[\frac{D_1}{A_3} + \frac{D_2\lambda_{21} - D_3\lambda_{22}}{\lambda_{21}^2 + \lambda_{22}^2}\right].$$

$$\text{and } \langle N_1(t)N_2(t)N_3(t)\rangle = F_1 + \frac{1}{T}[F_2 e^{A_3 T} + F_3 e^{2A_3 T} + F_4 e^{2\lambda_{21}T} + F_5 \cos\lambda_{22} t e^{\lambda_{21}T} + F_6 \sin\lambda_{22} t e^{\lambda_{21}T} +$$
$$F_7 \cos\lambda_{22} t e^{\lambda_{23}T} + F_8 \sin\lambda_{22} t e^{\lambda_{23}T} + F_9 \cos\lambda_{22} t e^{2\lambda_{21}T} + F_{10} \sin 2\lambda_{22} t e^{2\lambda_{21}T} - F_{11}]. \quad \textbf{(6.9)}$$

$$\text{Where; } F_1 = E_1, F_2 = \frac{E_2}{A_3}, F_3 = \frac{E_3}{2A_3}, F_4 = \frac{E_4}{2\lambda_{21}}, F_5 = \frac{E_5\lambda_{21} - E_6\lambda_{22}}{\lambda_{21}^2 + \lambda_{22}^2}, F_6 = \frac{E_5\lambda_{22} + E_6\lambda_{21}}{\lambda_{21}^2 + \lambda_{22}^2},$$

$$F_7 = \frac{E_7\lambda_{23} - E_8\lambda_{22}}{\lambda_{23}^2 + \lambda_{22}^2}, F_8 = \frac{E_7\lambda_{22} + E_8\lambda_{23}}{\lambda_{23}^2 + \lambda_{32}^2}, F_9 = \frac{E_9\lambda_{21} - E_{10}\lambda_{22}}{2(\lambda_{21}^2 + \lambda_{22}^2)}, F_{10} = \frac{E_9\lambda_{22} + E_{10}\lambda_{21}}{2(\lambda_{21}^2 + \lambda_{22}^2)},$$

$$\text{and } F_{11} = F_2 + F_3 + F_4 + F_5 + F_7 + F_9.$$

Since:

$$E_1 = d_4 D_4 D_1', E_2 = (d_1 D_4 + d_4 D_1)D_1', E_3 = d_1 D_1 D_1', E_4 = D_1'\left(\frac{d_2 D_2 + d_3 D_3}{2}\right),$$
$$E_5 = (d_2 D_4 + d_4 D_2)D_1', E_6 = (d_3 D_4 + d_4 D_3)D_1', E_7 = (d_1 D_2 + d_2 D_1)D_1', E_8 = (d_1 D_3 + d_3 D_1)D_1',$$

By definition of $G^{(2)}(t)$ we can calculation for $N_1(t)$, $N_2(t)$ and $N_3(t)$ and by definitions of $C_{ij}^{(4)}$ and $C_{ijk}^{(3)}$ $(i \neq J \neq k = 1,2,3)$ we can calculation for $N_i(t)\ N_j(t)$ and $N_i(t)\ N_J(t)\ N_k(t)$.

CONCLUSIONS

In this paper we considered two nonlinear systems one of them is of two equations and the other is of three equations, describing the evolution of rather complicated, the multispecies systems the iterative solutions are obtained where we are interested to study the Non-linear systems; so we achieved three requirements (i) study of equilibrium points and the associated stability, (ii) Analytical approximate solutions are obtained and lastly (iii) the correlation functions are computed. This study has been limited on the chronological correlation functions of the two type's; auto and cross. That is to reach the expense of second and third- correlation functions.

REFERENCES

[1].Lotka AJ: Elements of Physical Biology, New York, Williams and Wilkins, 1925.

[2].Volterra V: Variazioni e fluttuazioni Del numerod, individui in specie animali conviventi, Mem AcadLincei 3, 31-113, 1926.

[3].Murray JD: Mathematical biology, 2nd ed. New York, Springer, 1993.

[4].J. N. Kaput: Mathematical Models in Biology and Medicine, Affiliated East- West Press Private Limited New Delhi, 1992.

[5].J. N. Kapur: Stability Analysis of Continuous and Discrete Population Models, Indian J. Pure Appl. Math. , Vol 9 pp 702-08, 1976.

[6].J. N. Kapur: Moments for Some General Birth and Death Processes, Jour. Ind. Acad. Maths, Vol 1 pp 10-17,1979.

[7].J. N. Kapur, Umakumar: Generalised Birth and Death Processes with Twin Births, Nat Acad. Sci. Letters, Vol 1 pp 30-32,1978.

[8].J. N. Kapur: Predator-Prey Models with Discrete Time Lags, Nat. Acad. Sci. Letters, Vol 2 pp 237-75,1979.

[9].J. N. Kapur: Application of Generalized Hypergeometric Functions to Generalized Birth and Death Processes, Indian J. Pure Appl. Math. Vol 9 pp 1159- 69, 1978.

[10]. J. N. Kapur ,Q.J.A.kahan: Some Mathematical Models for Population Growth Indian J. PureAppl . Math., Vol 10 pp 277- 86, 1979.

[11]. J. N. Kapur, F. N. A: Population Dynamics via Games Theory and Modified Volterra Equations, Indian J. Pure Appl. Math, Vol 11 pp 347-53, 1980.

[12]. J. N. Kapur, F. N. A: Nonlinear Continuous-Time Discrete-Age-Scale Population Models, Indian J. Pure Appl. Math. Vol 11 pp 682-92, 1980.

[13]. J. N. Kapur: The Effect of Harvesting on Competing Populations, Mathematical Biosciences, Mathematical Biosciences, Vol 51 pp 175– 85,1980.

[14]. VlastimilkŘivan: Optimal Foraging and Predator-Prey Dynamics, Theoretical Population Biology 49,265-290,1996

[15]. Meng For, kewang: Periodicity in a Delayed Ratio-Detention Predator- Prey System, Journal of Mathematical Analysis and Applications 262, 179-190,2001.

[16]. Lin QIU, Taketomo Mitsui: Predator-Prey Dynamics with Delay when Prey Dispersing in n-Patch Environment, Graduate School of Human Informatics, Nagoya University, Japan, pp 1-14,2002.

[17]. Chris Flake, Tram Hoang, Elizabeth Perrigo: A Predator-Prey Model with Disease Dynamics, Under the Direction of Dr. Glenn Ledder, Department of Mathematics and Statistics, University of Nebraska- Lincoln, pp 116,2003.Paul Waltman, James Braselton, and Lorraine Braselton: A Mathematical Model of a Biological Arms Race with a Dangerous Prey, J. Theor Biol, 218, 55- 70,2002.
[18]. Ross Cressman, J. Garey: Evolutionary Stability in Lotka-Volterra Systems, J of Theoretical Biology, 222, 233-245, 2003.
[19]. Yan Ni Xiao, Lan sun Chen: Global Stability of a Predator-Prey System with Stage Structure for the Predator, Acta Mathematica Sinica, English Series, Vol. 19, No . 2, pp. 1-11, 2003.
[20]. PATRICK C. TOBIN, OTTAR N. BJØRNSTAD: Spatial Dynamics and Cross- Correlation in a Transient Predator-Prey System, Journal of Animal Ecology, 72, -467, 2003.
[21]. Volkan Sevim, Per Arne Rikvold: Effects of Correlated Interactions in a Biological Coevolution Model with Individual-Based Dynamics, J. Phys. A. math .Gen. 38, -9489, 2005.
[22]. A. F. Rozenfeld, C. J, Tessone, E. Albano and H. S .wio: On the Influesnce of Noise on the Critical and Oscillatory Behavior of a Predator-Prey Model: Coherent Stochastic Resonance at the Proper Frequency of the System, aleroz @ inifta. unlp. edu. ar, pp 1-18, (March 2006).
[23]. B. Batiha, M. S. M. Noorani, I. Hashim: Variational Iteration Method for Solving Multispecies Lotka-Volterra Equations, Computers and Mathematics with Applications 24, 903-909, 2007.
[24]. Jing Ruan, Yanhua Tan, Changsheng Zhang: A Modified Algorithm for Approximate Solutions of Lotka-Volterra Systems, Procedia Engineering 15, 1493-1497, 2011.
[25]. Nicola Serra: Utility Functions and Lotka-Volterra Model: A Possible Connection in Predator-Prey Game, Journal of Game Theory, 3(2), 31-34, 2014.
[26]. Susmita Paul, Sankar Prasad Mondal, Paritosh Bhattacharya: Numerical Solution of Lotka-Voiterra Prey Predator Model by Using Runge-Kutta-Fehlberg Method and Laplace Adomian Decomposition Method, Alexandria Engineering Journal 55, 613- 617, 2016.
[27]. YiğitAksoy, ÜnalGöktaş, Mehmet Pakdemirli, IhsanTimuçinDolapçı: Application of Perturbation-Iteration Method to Lotka-Volterra Equations, Alexandria Engineering Journal 55, 1661-1666, 2016.
[28]. Noura A. Abdulrazaq Tahar, Analysis of Hybrid Dynamical Systems with an Application in Biological Systems, Thesis of Master, Çankaya University,2017.