

THE (G'/G) -EXPANSION METHOD FOR SOLVING A NONLINEAR PDE DESCRIBING THE NONLINEAR LOW-PASS ELECTRICAL LINES

Khaled A. E. Alurffi^{1*}, Ayad M. Shahoot², Mohamed O. M. Elmridd³, Ali M. Almsiri³, and Abdullah M. H. Arwiniya³

^{*1} Department of Mathematics, Faculty of Science, Elmergib University, Khoms, Libya.

² Department of Physics, Faculty of Science, Elmergib University, Khoms, Libya.

³ Department of Mathematical Sciences, the Libyan Academy, Tripoli, Libya.

*Corresponding Author:-

E-mail: alurrfi@yahoo.com

Abstract:-

In this paper, we apply the (G'/G) -expansion method based on three auxiliary equations namely, the generalized Riccati equation $G'(\xi) = r + pG(\xi) + qG^2(\xi)$, the Jacobi elliptic equation $(G'(\xi))^2 = R + QG^2(\xi) + PG^4(\xi)$ and the second order linear ordinary differential equation (ODE) $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$ to find many new exact solutions of a nonlinear partial differential equation (PDE) describing the nonlinear low-pass electrical lines. The given nonlinear PDE has been derived and can be reduced to a nonlinear ordinary differential equation (ODE) using a simple transformation. Solitons wave solutions, periodic functions solutions, rational functions solutions and Jacobi elliptic functions solutions are obtained. Comparing our new solutions obtained in this paper with the well-known solutions are obtained. The given method in this paper is straightforward, concise and it can also be applied to other nonlinear PDEs in mathematical physics.

Keywords:- The (G'/G) -expansion method; Exact solutions; Solitons wave solutions; Periodic solutions; the Generalized Riccati equation; Jacobi elliptic functions solutions.

INTRODUCTION

In the recent years, investigations of exact solutions to nonlinear PDEs play an important role in the study of nonlinear physical phenomena in such as fluid mechanics, hydrodynamics, optics, plasma physics, solid state physics, and biology. Several methods for finding the exact solutions to nonlinear equations in mathematical physics have been presented, such as the inverse scattering method [1], the Hirota bilinear transform method [2], the truncated Painlevé expansion method [3,4], the Bäcklund transform method [5,6], the exp-function method [7-9], the tanh-function method [10-12], the Jacobi elliptic function expansion method [13-15], the (G'/G) -expansion method [16-20], the $(G'/G, 1/G)$ -expansion method [21-23], the generalized projective Riccati equations method [24-26] and so on.

The objective of this paper is to use the (G'/G) -expansion method with the aid of Computer algebraic system Maple to construct many exact solutions of the following nonlinear PDE governing wave propagation in nonlinear low-pass electrical transmission lines [15,27]:

$$\frac{\partial^2 V(x,t)}{\partial t^2} - \alpha \frac{\partial^2 V^2(x,t)}{\partial t^2} + \beta \frac{\partial^2 V^3(x,t)}{\partial t^2} - \delta^2 \frac{\partial^2 V(x,t)}{\partial x^2} - \frac{\delta^4}{12} \frac{\partial^4 V(x,t)}{\partial x^4} = 0, \quad (1.1)$$

Where α , β and δ are constants, while $V(x,t)$ is the voltage in the transmission lines. The variable x is interpreted as the propagation distance and t is the slow time. The physical details of the derivation of Eq. (1.1) using the Kirchhoff's laws given in [27]. Eq. (1.1) has been discussed in [15, 27] using a new Jacobi elliptic function expansion method and an auxiliary equation method respectively, and its exact solutions have been found. This paper is organized as follows: In Sec. 2, the description of the (G'/G) -expansion method is given. In Sec. 3, we use the given method described in Sec. 2, to find exact solutions of Eq. (1.1). In Sec. 4, physical explanations of some results are presented. In Sec. 5, some conclusions are obtained

DESCRIPTION OF THE (G'/G) -EXPANSION METHOD

Consider a nonlinear PDE in the form:

$$P(V, V_x, V_t, V_{xx}, V_{xt}, \dots) = 0, \quad (2.1)$$

where $V = V(x,t)$ is a unknown function, P is a polynomial in $V = V(x,t)$ and its Partial derivatives in which the highest order derivatives and nonlinear terms are involved.

Let us now give the main steps of the (G'/G) -expansion method [16-20]:

Step 1. We look for the voltage $V(x,t)$ in the traveling form:

$$V(x,t) = V(\xi), \quad \xi = \sqrt{k}(x - \omega t), \quad (2.2)$$

Where k and ω are undetermined positive parameters, and ω is the velocity of Propagation, to reduce Eq. (2.1) to the following nonlinear ODE

$$H(V, V', V'', \dots) = 0, \quad (2.3)$$

where H is a polynomial of $V(\xi)$ and its total derivatives $V'(\xi), V''(\xi), \dots$ and $' = \frac{d}{d\xi}$.

Step 2. We assume that the solution of Eq. (2.3) has the form:

$$V(\xi) = \sum_{i=0}^N a_i \left(\frac{G'}{G} \right)^i \quad (2.4)$$

Where a_i ($i = 1, 2, \dots, N$) constants to be determined later, provided $a_N \neq 0$ and $G = G(\xi)$ satisfies the following three auxiliary equations:

1) The generalized Riccati equation

$$G'(\xi) = r + pG(\xi) + qG^2(\xi), \quad (2.5)$$

2) The Jacobi elliptic equation

$$(G'(\xi))^2 = R + QG^2(\xi) + PG^4(\xi), \quad (2.6)$$

3) The second order linear ODE

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (2.7)$$

Where $r, p, q, R, Q, P, \lambda$ and μ are constants to be determined later?

Step 3. We determine the positive integer N in (2.4) by balancing the highest-order Derivatives and the highest nonlinear terms in Eq. (2.3)

Step 4. Substituting (2.4) along with Eqs. (2.5)- (2.7) into Eq. (2.3) and collecting all the coefficients of $G'(\xi), (i=0, \pm 1, \pm 2, \dots)$ for Eq.(2.5), $\left(\frac{G'(\xi)}{G(\xi)}\right), (i=0, 1, 2, \dots)$ for Eqs.(2.6) and (2.7), then setting these coefficients to zero, yield a set of algebraic equations, which can be solved by using the Maple or Mathematica to find the values of $a_i (i=0, 1, 2, \dots, N), r, p, q, R, Q, P, \lambda$ and μ .

Step 5. It is well-known that Eqs. (2.5)-(2.7) have many families of solutions obtained in [16-20].

Step 6. Substituting the values of $a_i (i=1, 2, \dots, N), r, p, q, R, Q, P, \lambda$ and μ as well as the solutions of step 5 into (2.4) we have the exact solutions of Eq. (2.1)

On solving Eq. (1.1) using the proposed method of Sec. 2

In this section, we apply the $\left(\frac{G'}{G}\right)$ expansion method of Sec. 2 to find new exact solutions of Eq. (1.1). To this aim, we use the transformation (2.2) to reduce Eq. (1.1) to the following nonlinear ODE:

$$\frac{d^2}{d\xi^2} \left\{ \frac{k^2 \delta^4}{12} \frac{d^3 V}{d\xi^2} + (k \delta^2 - k \omega^2) V + \alpha k \omega^2 V^2 - \beta k \omega^2 V^3 \right\} = 0. \quad (3.1)$$

Integrating Eq. (3.1) with respect to ξ twice, and vanishing the constants of integration, we find the following ODE:

$$\frac{K^2}{12} \frac{d^3 V}{d\xi^2} + (K - U) V + \alpha U V^2 - \beta U V^3 = 0. \quad (3.2)$$

where $K = k \delta^2$ and $U = k \omega^2$.

Balancing $\frac{d^3 V}{d\xi^2}$ with V^3 gives $N=1$. Therefore, (2.4) reduces to

$$V(\xi) = a_0 + a_1 \left(\frac{G'}{G} \right), \quad (3.3)$$

Where a_0 and a_1 are constants to be determined such that $a_1 \neq 0$.

Exact solutions of Eq. (1.1) depending on the Riccati equation (2.5)

In this subsection, substituting (3.3) along with the generalized Riccati equation (2.5) into Eq. (3.2) and collecting all the coefficients of $G'(\xi), (i=0, \pm 1, \pm 2, \pm 3, \pm 4)$ and setting them to zero, we get a system of algebraic equations for $a_i (i=0, 1), r, p, q, U,$ and K . Using the Maple or Mathematica, we get the following results:

Result 1.

$$K = \frac{-24\alpha^2}{p^2(2\alpha^2 - 9\beta)}, U = \frac{216\alpha^2\beta}{p^2(2\alpha^2 - 9\beta)^2}, a_0 = 0, a_1 = \frac{2\alpha}{3p\beta}, r = 0. \quad (3.1.1)$$

Substituting (3.1.1) into (3.3) yields

$$V(\xi) = \frac{2\alpha}{3p\beta} \left(\frac{G'}{G} \right), \quad (3.1.2)$$

$$\xi = \sqrt{\frac{-24\alpha^2}{p^2\delta^2(2\alpha^2 - 9\beta)}} x - \sqrt{\frac{216\alpha^2\beta}{p^2(2\alpha^2 - 9\beta)^2}} t, \quad 2\alpha^2 < 9\beta, \beta > 0 \text{ and } p \neq 0.$$

where

With reference to solving Eq. (2.7) [19], we deduce that the exact solutions of Eq. (1.1) as follows:

$$V(\xi) = \frac{2\alpha}{3\beta} \left(\frac{\cosh(p\xi) - \sinh(p\xi)}{d + \cosh(p\xi) - \sinh(p\xi)} \right), \quad (3.1.3)$$

$$V(\xi) = \frac{2\alpha}{3\beta} \left(\frac{d}{d + \cosh(p\xi) + \sinh(p\xi)} \right), \quad (3.1.4)$$

RESULTS

$$K = \frac{-24\alpha^2}{2\alpha^2(p^2 - 4qr) - 9\beta p^2}, \quad U = \frac{216\alpha^2\beta}{(2\alpha^2(p^2 - 4qr) - 9\beta p^2)^2}, \quad a_0 = 0, \quad a_1 = \frac{2\alpha}{3p\beta}. \quad (3.1.5)$$

Substituting (3.1.5) into (3.3) yields

$$V(\xi) = \frac{2\alpha}{3p\beta} \left(\frac{G'}{G} \right),$$

(3.1.6)

$$\text{where } \xi = \sqrt{\frac{-24\alpha^2}{2\alpha^2\delta^2(p^2 - 4qr) - 9\beta p^2}} x - \sqrt{\frac{216\alpha^2\beta}{(2\alpha^2(p^2 - 4qr) - 9\beta p^2)^2}} t, \\ 2\alpha^2\delta^2(p^2 - 4qr) < 9\beta p^2 \text{ and } \beta > 0.$$

In this result, we deduce that the exact solutions of Eq. (1.1) as follows

$$V(\xi) = \frac{\alpha}{3\beta p} \left(\frac{(p^2 - 4qr) \operatorname{sech}^2\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right)}{p + \sqrt{p^2 - 4qr} \tanh\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right)} \right), \quad (3.1.7)$$

$$V(\xi) = -\frac{\alpha}{3\beta p} \left(\frac{(p^2 - 4qr) \operatorname{csch}^2\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right)}{p + \sqrt{p^2 - 4qr} \coth\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right)} \right), \quad (3.1.8)$$

$$V(\xi) = -\frac{2\alpha(p^2 - 4qr)}{3\beta p} \left(\frac{\operatorname{csch}^2\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right) + \operatorname{csch}\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right) \coth\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right)}{p + \sqrt{p^2 - 4qr} \left(\coth\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right) + \operatorname{csch}\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right) \right)} \right), \quad (3.1.9)$$

$$V(\xi) = -\frac{\alpha(p^2 - 4qr)}{3p\beta \cosh\left(\frac{1}{2}\sqrt{-p^2 + 4qr}\xi\right)} \\ \times \left(\frac{1}{\left(\sqrt{p^2 - 4qr} \sinh\left(\frac{1}{2}\sqrt{p^2 - 4qr}\xi\right) - p \cosh\left(\frac{1}{2}\sqrt{p^2 - 4qr}\xi\right) \right)} \right), \quad (3.1.10)$$

$$V(\xi) = \frac{\alpha(p^2 - 4qr)}{3p\beta \sinh\left(\frac{1}{2}\sqrt{-p^2 + 4qr}\xi\right)} \\ \times \left(\frac{1}{\left(\sqrt{p^2 - 4qr} \cosh\left(\frac{1}{2}\sqrt{p^2 - 4qr}\xi\right) - p \sinh\left(\frac{1}{2}\sqrt{p^2 - 4qr}\xi\right) \right)} \right), \quad (3.1.11)$$

where $p^2 - 4qr > 0$.

$$V(\xi) = \frac{1}{3} \frac{\alpha(p^2 - 4qr) \left(1 + \tan^2 \left(\frac{1}{2} \sqrt{-p^2 + 4qr} \xi \right) \right)}{p\beta \left(-\sqrt{-p^2 + 4qr} \tan \left(\frac{1}{2} \sqrt{-p^2 + 4qr} \xi \right) - p \right)}, \quad (3.1.12)$$

$$V(\xi) = \frac{1}{3} \frac{\alpha(p^2 - 4qr) \left(\cot^2 \left(\frac{1}{2} \sqrt{-p^2 + 4qr} \xi \right) + 1 \right)}{p\beta \left(p + \sqrt{-p^2 + 4qr} \cot \left(\frac{1}{2} \sqrt{-p^2 + 4qr} \xi \right) \right)}, \quad (3.1.13)$$

$$V(\xi) = \frac{\alpha(p^2 - 4qr)}{3p\beta \cos \left(\frac{1}{2} \sqrt{-p^2 + 4qr} \xi \right)} \times \left(\frac{1}{\sqrt{-p^2 + 4qr} \sin \left(\frac{1}{2} \sqrt{-p^2 + 4qr} \xi \right) + p \cos \left(\frac{1}{2} \sqrt{-p^2 + 4qr} \xi \right)} \right), \quad (3.1.14)$$

where $p^2 - 4qr > 0$.

$$V(\xi) = \frac{2}{3} \frac{\alpha(-\sinh(p\xi) + \cosh(p\xi))}{\beta(d + \cosh(p\xi) - \sinh(p\xi))}, \quad (3.1.15)$$

$$V(\xi) = \frac{2}{3} \frac{\alpha d}{(\sinh(p\xi) + \cosh(p\xi))}. \quad (3.1.16)$$

Also, there are many other exact solutions of Eq. (1.1), which are omitted here for simplicity.

Exact solutions of Eq. (1.1) depending on the Jacobi elliptic equation (2.6)

In this subsection, substituting (3.3) along with the Jacobi elliptic equation (2.6) into Eq. (3.2) and collecting all the

coefficients of $\left(\frac{G'(\xi)}{G(\xi)} \right)^i$, ($i = 0, 1, 2, \dots$) and setting them to be zero, we have the following algebraic equations:

$$\left(\frac{G'(\xi)}{G(\xi)} \right)^3 : \frac{1}{6} K^2 a_1 - \beta U a_1^3 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)} \right)^2 : -3\beta U a_0 a_1^2 + \alpha U a_1^2 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)} \right) : \frac{-1}{6} K^2 a_0 Q + (K - U) a_1 + 2\alpha U a_0 a_1 - 3\beta U a_0^2 a_1 = 0,$$

$$\left(\frac{G'(\xi)}{G(\xi)} \right)^0 : (K - U) a_0 + \alpha U a_0^2 - \beta U a_0^3 = 0. \quad (3.2.1)$$

On solving the above algebraic equations (3.2.1) using the Maple or Mathematical, we have the following result:

$$K = -\frac{6\alpha^2}{Q(2\alpha^2 - 9\beta)}, U = \frac{54\alpha^2\beta}{Q(2\alpha^2 - 9\beta)^2}, a_0 = -\frac{\alpha}{3\beta}, a_1 = \pm \frac{\alpha}{3\beta\sqrt{Q}}, P = P, R = R, \quad (3.2.2)$$

where $Q > 0$.

Substituting (3.2.2) into (3.3) yields

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \frac{1}{\sqrt{Q}} \left(\frac{G'(\xi)}{G(\xi)} \right) \right], \quad (3.2.3)$$

$$\text{where } \xi = \sqrt{-\frac{6\alpha^2}{Q(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{Q(2\alpha^2-9\beta)^2}}t, \quad 2\alpha^2 < 9\beta, \beta > 0.$$

With reference to solving Eq. (2.6) [20], we deduce that the Jacobi elliptic functions solutions and other exact solutions of Eq. (1.1) as follows:

Case 1. Choosing $P = -m^2$, $Q = 2m^2 - 1$, $R = 1 - m^2$, and $G(\xi) = \text{cn}(\xi)$, we obtain The Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \frac{\text{sc}(\xi) \text{dn}(\xi)}{\sqrt{2m^2 - 1}} \right], \quad (3.2.4)$$

$$\text{where } \xi = \sqrt{-\frac{6\alpha^2}{(2m^2-1)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2m^2-1)(2\alpha^2-9\beta)^2}}t.$$

If $m \rightarrow 1$, then Eq. (1.1) has the kink soliton wave solutions

$$V(\xi) = \frac{\alpha}{3\beta} [1 \pm \tanh(\xi)], \quad (3.2.5)$$

$$\text{where } \xi = \sqrt{-\frac{6\alpha^2}{(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2\alpha^2-9\beta)^2}}t.$$

Case 2. Choosing $P = -1$, $Q = 2 - m^2$, $R = m^2 - 1$, and $G(\xi) = \text{dn}(\xi)$, we obtain the The Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \frac{m^2 \text{cd}(\xi) \text{sn}(\xi)}{\sqrt{2 - m^2}} \right], \quad (3.2.6)$$

$$\text{where } \xi = \sqrt{-\frac{6\alpha^2}{(2-m^2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2-m^2)(2\alpha^2-9\beta)^2}}t$$

If $m \rightarrow 1$, then we have the same exact solutions Eq. (3.2.5).

Case 3. Choosing $P = 1 - m^2$, $Q = 2 - m^2$, $R = 1$, and $G(\xi) = \text{sc}(\xi)$, we obtain the the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \frac{\text{ds}(\xi)}{\text{cn}(\xi)\sqrt{2 - m^2}} \right], \quad (3.2.7)$$

$$\text{where } \xi = \sqrt{-\frac{6\alpha^2}{(2-m^2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2-m^2)(2\alpha^2-9\beta)^2}}t.$$

If $m \rightarrow 0$, then Eq. (1.1) has the trigonometric solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \frac{\sec(\xi) \csc(\xi)}{\sqrt{2}} \right], \quad (3.2.8)$$

$$\text{where } \xi = \sqrt{-\frac{3\alpha^2}{(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{27\alpha^2\beta}{(2\alpha^2-9\beta)^2}}t.$$

If $m \rightarrow 1$, then Eq. (1.1) has the anti-kink soliton wave solutions

$$V(\xi) = \frac{\alpha}{3\beta} [1 \pm \coth(\xi)], \quad (3.2.9)$$

$$\text{where } \xi = \sqrt{-\frac{6\alpha^2}{(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2\alpha^2-9\beta)^2}}t.$$

Case 4. Choosing $P = \frac{1}{4}$, $Q = \frac{1-2m^2}{2}$, $R = \frac{1}{4}$, and $G(\xi) = \text{ns}(\xi) \pm \text{cs}(\xi)$, we obtain the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \frac{\sqrt{2} \text{ds}(\xi)}{\sqrt{1-2m^2}} \right], \quad (3.2.10)$$

$$\text{where } \xi = \sqrt{-\frac{12\alpha^2}{(1-2m^2)(2\alpha^2-9\beta)}\delta^2}x - \sqrt{\frac{108\alpha^2\beta}{(1-2m^2)(2\alpha^2-9\beta)^2}}t$$

If $m \rightarrow 0$, then Eq. (1.1) has the trigonometric solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \sqrt{2} \csc(\xi) \right], \quad (3.2.11)$$

$$\text{where } \xi = \sqrt{-\frac{12\alpha^2}{(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{108\alpha^2\beta}{(2\alpha^2-9\beta)^2}}t.$$

Case 5. Choosing $P = -1$, $Q = 2-m^2$, $R = m^2-1$, and $G(\xi) = \text{nd}(\xi)$, we obtain the the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \frac{m^2 \text{cn}(\xi) \text{sd}(\xi)}{\sqrt{2-m^2}} \right], \quad (3.2.12)$$

$$\text{where } \xi = \sqrt{-\frac{6\alpha^2}{(2-m^2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2-m^2)(2\alpha^2-9\beta)^2}}t.$$

If $m \rightarrow 1$, then we have the same exact solutions Eq. (3.2.5).

Case 6. Choosing $P = 1-m^2$, $Q = 2-m^2$, $R = 1$, and $G(\xi) = \text{cs}(\xi)$, we obtain the The Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \frac{\text{nc}(\xi) \text{ds}(\xi)}{\sqrt{2-m^2}} \right], \quad (3.2.13)$$

$$\text{where } \xi = \sqrt{-\frac{6\alpha^2}{(2-m^2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2-m^2)(2\alpha^2-9\beta)^2}}t.$$

If $0 \rightarrow m$, then we have the same exact solutions Eq. (3.2.8).

If $1 \rightarrow m$, then we have the same exact solutions Eq. (3.2.9).

Case 7. Choosing $P = m^2(m^2-1)$, $Q = 2m^2-1$, $R = 1$, and $G(\xi) = \text{ds}(\xi)$, we obtain the Jacobi elliptic function Solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \frac{\text{cd}(\xi) \text{ns}(\xi)}{\sqrt{2m^2-1}} \right], \quad (3.2.14)$$

$$\text{where } \xi = \sqrt{-\frac{6\alpha^2}{(2m^2-1)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2m^2-1)(2\alpha^2-9\beta)^2}}t.$$

If $m \rightarrow 1$, then we have the same exact solutions Eq. (3.2.9).

Case 8. Choosing $P = (m^2-1)$, $Q = 2-m^2$, $R = -1$, and $G(\xi) = \text{nd}(\xi)$, we obtain the The Jacobi elliptic function Solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \frac{m^2 \text{cd}(\xi) \text{sd}(\xi) \text{dn}(\xi)}{\sqrt{2-m^2}} \right], \quad (3.2.15)$$

$$\text{where } \xi = \sqrt{-\frac{6\alpha^2}{(2-m^2)(2\alpha^2-9\beta)\delta^2}}x - \sqrt{\frac{54\alpha^2\beta}{(2-m^2)(2\alpha^2-9\beta)^2}}t.$$

If $m \rightarrow 1$, then we have the same exact solutions Eq. (3.2.5).

Case 9. Choosing $P = \frac{m^2-1}{4}$, $Q = \frac{m^2+1}{2}$, $R = \frac{m^2-1}{4}$, and $G(\xi) = m \text{sd}(\xi) \pm \text{nd}(\xi)$, we The Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \frac{m\sqrt{2}\operatorname{cd}(\xi)}{\sqrt{1+m^2}} \right], \quad (3.2.16)$$

$$\text{where } \xi = \sqrt{-\frac{12\alpha^2}{(m^2+1)(2\alpha^2-9\beta)}\delta^2}x - \sqrt{\frac{108\alpha^2\beta}{(m^2+1)(2\alpha^2-9\beta)^2}}t.$$

If $m \rightarrow 1$, then we have the trivial solution.

Case 10. Choosing $P = \frac{1}{4}$, $Q = \frac{1-2m^2}{2}$, $R = \frac{1}{4}$, and $G(\xi) = \operatorname{ns}(\xi) \pm \operatorname{cs}(\xi)$, we obtain the the Jacobi elliptic function solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \frac{\sqrt{2}\operatorname{ds}(\xi)}{\sqrt{1-2m^2}} \right], \quad (3.2.17)$$

$$\text{where } \xi = \sqrt{-\frac{12\alpha^2}{(1-2m^2)(2\alpha^2-9\beta)}\delta^2}x - \sqrt{\frac{108\alpha^2\beta}{(1-2m^2)(2\alpha^2-9\beta)^2}}t.$$

If $m \rightarrow 0$, then we have the same exact solutions Eq. (3.2.11).

Exact solutions of Eq. (1.1) depending on the second order linear ODE (2.7)

Here, Substituting (3.3) along with the second order linear ODE (2.7) into Eq. (3.2) and collecting all the coefficients of

$\left(\frac{G'(\xi)}{G(\xi)}\right)^i$, ($i = 0, 1, 2, 3$) And setting them to be zero, we have the following algebraic equations:

$$\begin{aligned} \left(\frac{G'(\xi)}{G(\xi)}\right)^3 &: \frac{1}{6}K^2a_1 - U\beta a_1^3 = 0, \\ \left(\frac{G'(\xi)}{G(\xi)}\right)^2 &: \frac{1}{4}K^2\lambda a_1 + U\alpha a_1^2 - 3U\beta a_0 a_1^2 = 0, \\ \left(\frac{G'(\xi)}{G(\xi)}\right) &: \frac{1}{12}K^2a_1(\lambda^2 + 2\mu) + (K - U)a_1 + 2U\alpha a_0 a_1 - 3U\beta a_0^2 a_1 = 0, \\ \left(\frac{G'(\xi)}{G(\xi)}\right)^0 &: \frac{1}{12}K^2\lambda\mu a_1 + (K - U)a_0 + U\alpha a_0^2 - U\beta a_0^3 = 0. \end{aligned} \quad (3.3.1)$$

On solving the above algebraic equations (3.3.1) using the Maple or Mathematical, we have the following result:

$$\begin{aligned} K &= -\frac{24\alpha^2}{(\lambda^2 - 4\mu)(2\alpha^2 - 9\beta)}, U = \frac{216\alpha^2\beta}{(\lambda^2 - 4\mu)(2\alpha^2 - 9\beta)^2}, a_0 = \frac{\alpha}{3\beta} \left(1 \pm \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} \right), \\ a_1 &= \pm \frac{2\alpha}{3\beta\sqrt{\lambda^2 - 4\mu}}. \end{aligned} \quad (3.3.2)$$

Substituting (3.3.2) into (3.3) yields

$$V(\xi) = \frac{\alpha}{3\beta} \left(1 \pm \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} \right) \pm \frac{2\alpha}{3\beta\sqrt{\lambda^2 - 4\mu}} \left(\frac{G'(\xi)}{G(\xi)} \right), \quad (3.3.3)$$

$$\xi = \sqrt{-\frac{24\alpha^2}{\delta^2(\lambda^2 - 4\mu)(2\alpha^2 - 9\beta)}}x - \sqrt{\frac{216\alpha^2\beta}{(\lambda^2 - 4\mu)(2\alpha^2 - 9\beta)^2}}t, \quad 2\alpha^2 < 9\beta, \beta > 0, \text{ and } \lambda^2 - 4\mu \neq 0.$$

With reference to solving Eq. (2.7) [16-18], we deduce that the hyperbolic functions solutions and the trigonometric functions solutions as follows

Case 1. If $\lambda^2 - 4\mu > 0$, then we have the hyperbolic functions solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \frac{A_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi\right) + A_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi\right)}{A_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi\right) + A_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi\right)} \right], \quad (3.3.4)$$

In particular, if we set $A_1 \neq 0, A_2 = 0, \lambda > 0$ and $\mu = 0$, in (3.3.4), then we get the kink soliton wave solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \tanh\left(\frac{\lambda}{2}\xi\right) \right], \quad (3.3.5)$$

while, if we set $A_1 = 0, A_2 \neq 0, \lambda > 0$ and $\mu = 0$, then we get the anti-kink soliton wave solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm \coth\left(\frac{\lambda}{2}\xi\right) \right], \quad (3.3.6)$$

$$\text{where } \xi = \sqrt{-\frac{24\alpha^2}{\delta^2\lambda^2(2\alpha^2 - 9\beta)}}x - \sqrt{\frac{216\alpha^2\beta}{\lambda^2(2\alpha^2 - 9\beta)^2}}t.$$

Case 2. If $\lambda^2 - 4\mu < 0$, then we have the trigonometric functions solutions

$$V(\xi) = \frac{\alpha}{3\beta} \left[1 \pm i \frac{A_1 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right) - A_2 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right)}{A_1 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right) + A_2 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right)} \right], i = \sqrt{-1} \quad (3.3.7)$$

In particular, if we set $A_1 = 0, A_2 \neq 0, \lambda > 0$ and $\mu = 0$, in (3.3.7), Then we get the same kink solution wave solutions (3.3.5), while, if we set $A_1 \neq 0, A_2 = 0, \lambda > 0$ and $\mu = 0$, then we get the same anti-kink solution wave solutions.

PHYSICAL EXPLANATIONS OF SOME RESULTS

In this section, we have presented some graphs of the exact solutions. These solutions are solution solutions, periodic solutions and Jacobi elliptic functions solutions. Exact solutions of the results describe different nonlinear waves. For the established exact solutions with hyperbolic solutions are special kinds of solitary wave's solutions. These solutions have a remarkable property that keeps its identity upon interacting with other. Let us now examine Figs. 1-3 as it illustrates some of our solutions obtained in this paper. To this aim, we select some special values of the parameters obtained, for example, in some of the solutions (3.1.3), (3.1.7) and (3.2.4) of the nonlinear PDE (1.1). For more convenience the graphical representations of these solutions are shown in the following figures:

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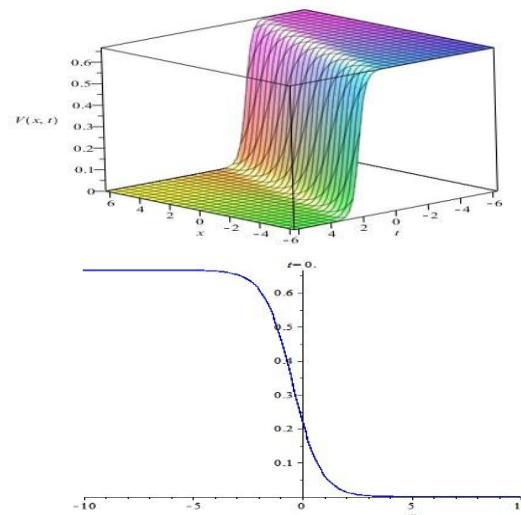


Fig. 1 The plot of solution (3.1.3) when $a=2, b=2, \delta=2, p=1, d=2$.

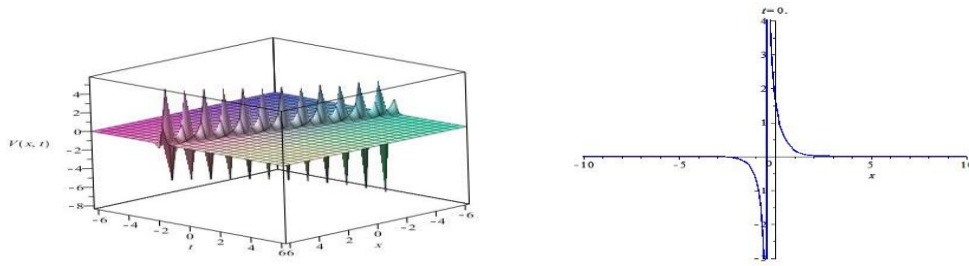


Fig. 2 The plot of solution (3.1.7) when $\alpha=2, \beta=2, \delta=3, p=d=q=1, r=-1$.

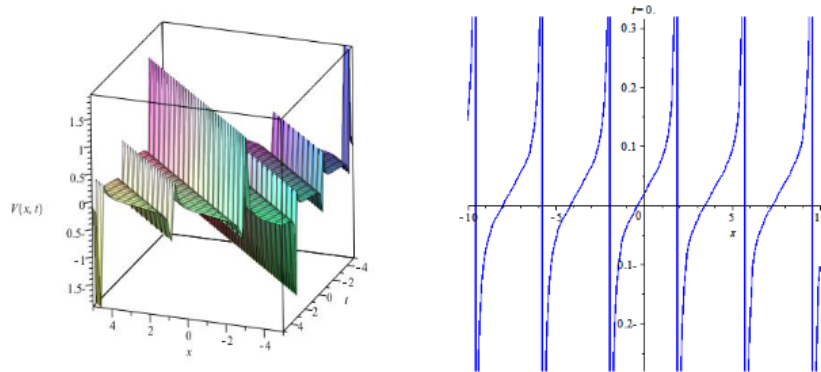


Fig. 3 the plot of solution (3.2.4) when $\alpha=3, \beta=26+4\sqrt{42}, \delta=\frac{2\sqrt{18+3\sqrt{42}}}{6+\sqrt{42}}, m=\frac{3}{4}$.

CONCLUSIONS

In this paper, we have solved the nonlinear PDE describing the nonlinear low-pass electrical transmission lines (1.1) using the (G'/G) -expansion method with the aid of three auxiliary equations (2.5)-(2.6) described in Sec. 2. By the aid of Maple or Mathematica, we have found many solutions of Eq. (1.1) which are new. On comparing our results with the results obtain in [15, 27] using the new Jacobi elliptic function expansion method and the auxiliary equation method respectively, we deduce that our results are different and new. Also, we have noted that our results (3.3.5) and (3.3.6) are in agreement with the results (3.2.5) and (3.2.9) of Sec 3.2. Obtained in this paper respectively, when $\lambda=2$. Further, all solutions obtained in this paper have been checked with the Maple by putting them back into the original equations. Finally, the proposed method in this paper can be applied to many other nonlinear PDEs in mathematical physics.

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