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ON CLOSED FULLY STABLE ACTS

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Abstract:
The purpose of this paper is to introduce and study closed fully stable acts as a concept of generalization of fully stable acts. Some properties and characterizations of class of closed fully stable acts are considered .The relations between this class and other well classes of acts, like quasi-injective acts and other classes of injectivityare discussed

Keywords:-*Closed fully stable act, extending act, quasi-injective act, closed fully pseudo act.*

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1-INTRODUCTION

Throughout this work, S is a monoid with zero element and every S-act is unitary right S-act with zero element Θ which denoted by Ms. For more details about S-acts we refer the reader to the reference [1].M.S. Abbas introduced in [2] a class of modules which was called a fully stable module which prompted Hiba to give the corresponding definition for S-acts as follows: let Ms be an S-act. A subact N of Msis called stable, if f(N)⊆N for each S-homomorphism f:N⟶Ms. An S-act M is called fully stable in case each subact of Ms is stable. A monoid S is fully stable if it is a fully stable S-act[3].A subsystem N of S-system Ms is called closed if it has no proper ⋂-large in Ms that is the only solution of N ↪⋂lL↪≠Ms is N = L [4].The concept of fully stable S-act motivated us to introduce and study a generalization of its which is closed fully stable concept as a class of acts, and give several characterizations of the seacts.A part of this paper devoted to study the relations between this class and some acts like quasi-injective, Baer's criterion and extending acts.

2-Closed fully stable acts:

Definition (2.1):

Let Msbe a right S-act. A closed subact N of Ms is called closed stable if f(N) ⊆N for each S-homomorphism f :N ⟶Ms. An S-act Ms is called closed fully stable act (for short cl-fully stable) in case each closed subact of Ms is stable . A monoid S is closed fully stable if it is cl-fully stable S-act .

Remarks and Examples (2.2):

1-Every fully stable act is cl-fully stable act , but the converse is not true in general for example Z with multiplication as Z-act is cl-fully stable act but not fully stable

2-Isomorphism act to cl-fully stable act is cl-fully stable act

3-Everycl-fully stable is fully invariant, but the converse is not true in general for example Z with multiplication as Z-act is cl-fully stable act but not fully invariant , for this if we define f: 2Z ⟶Z by f(2n)=3n , then it is clear that f(2Z) ⊈2Z since f(2.1)=3 ∉f(2Z).

The following lemmas explain under which conditions the subact inherit the property of closed fully stable act:

Lemma(2.4): Every closed subact of closed fully stable act is closed fullystable.Proof: Let Msbe closed fully stable S-act and N be closed subact of Ms. Let H be closed subact of N. Then His closed subact of Msby lemma (2.4) in [5]. Let f:H⟶N be an S-homomorphism and iN:N⟶Msbe the inclusion map ,so iN∘f:H⟶Msbe an S-homomorphism. As Msis cl-fully stable act, so iN∘f(H)⊆H and this implies that f(H)⊆H . Thus N is cl-fully stableact.

Lemma (2.5): Every retract subact of closed fully stable act is closed fully stable.

Proof: By remarks and examples (2.2) (4) in [4] and by lemma (2.4).

Proposition(2.6): Let Msbe an S-act in which every closed is a retract of Ms. If End(Ms) is commutative , then Msis cl-fully stable act.

Proof: Let N be any closed subact of Msand f: N⟶Msbe an S-homomorphism. Then, by assumption there exists a subact H such that Ms=N⋃̇H. f can be extended to an S-homomorphism g: Ms⟶Msby putting g(h)=Θfor each h ∈H. Define K: Ms⟶Ms by K(x,y) = x for each x ∈N and y∈H. Let f(x) =(y, 1) for some y ∈N and 1∈H. Now K∘g(w)=K(g(x,h))=K(g(x))=K(f(x))=K(y,1)=y . On the other hand gun (w) =g (K(x,y))=g(x)=f(x)=(y,1). Since K∘g = g∘K, then (y,1) = (y,0) and 1= 0 which is a contradiction. Thus f(x) ∈N and therefore f(N) ⊆N , hence Msis cl-fully stable .Because in extending acts every closed subact is a retract in[5],then we have:

Corollary (2.7): Let Msbe extending act. Then Msis cl-fully stable if and only if End(Ms) is commutative

Proposition(2.8): Let Msbe an S-act such that every closed subact is a retract of Ms. If End (Ms) is cl-fully stable monoid, then Msis cl-fully stable act.

Proof: Let N be closed subact of Msα: N ⟶Ms. Consider K = Hom(Ms, N) , is closedright ideal of End(Ms) . Define β: K ⟶End (Ms) by β (f) = α∘f for each f ∈K. Clearly, β(f) ∈End(Ms) , moreover βis End(Ms)-homomorphism Since End(Ms) is cl-fully stable , so β(K) ⊆K . That is for each f ∈K, α∘f ∈K and then α∘f: Ms⟶N. But N is a retract of Ms, then the natural projection πNof Msonto N is in K , hence α∘πN∈K . That is α∘πN: Ms⟶N, since πNis onto, so α: N ⟶N or α (N) ⊆N. Thus Msis cl-fully stable act. ■

Corollary (2.9): Let Msbe extending S-act. If End(Ms) is cl-fully stable monoid , then Msis cl-fully stable act

Proposition(2.10): Let Msbe S-act such that every subactis closed. Then Msis cl-fully stable act if and only if End(Ms) is cl-fully stable monoid .

Proof: Let Msbe cl-fully stable and extending S-act. Let I= Hom(Ms,N) be closed right ideal of End(Ms)andα:I⟶End(Ms) . AsMsis cl-fully stable, sofor each S-homomorphism f: N⟶Ms,f(N)⊆N. Then, it is clear thatfor each g ∈I,f∘g ∈End(Ms).Since End(Ms) is commutative by corollary(2.7) ,so f∘g=g∘f . This means that f,g are isomorphisms . Then, sincef: N⟶N, so we havef∘g ∈I. This implies that f∘g ∈End(Ms) = α(I) and on the other hand f∘g ∈I . Therefore, α (I) ⊆I.

Corollary (2.11): Let Msbe quasi injective S-act. Then Msis cl-fully stable act if and only if End(Ms) is cl-fully stable monoid .

Corollary (2.12): Let Msbe quasi injective S-act with ψM= I. Then the following statements are equivalent:

1-Msis cl-fully stable act;

2-End (Ms) is commutative monoid;

3-End (Ms) is cl-fully stable act.

Proof: (1⟶2) As quasi injective S-act with ψM= i is extending act by proposition(4.1)in[5], so by corollary(2.7) End(Ms) is commutative monoid .

(2⟶3) As pervious argument in (1⟶2), we obtainMsis extending, so by corollary(2.7) ,Msis cl-fully stable act and then by corollary(2.11) implies that End(Ms) is cl-fully stable act .
(1⟶2) By corollary (2.11).
The following proposition explain the characterization of closed stable subact :
Proposition(2.13): Let Msbe an S-act and Ms= A ⋃B, where A and B are two subacts of Ms. If N is closed stable subact of Ms, then N = (A⋂N) ⋃(B⋂N) .
Proof: Let πA: Ms⟶A and πB: Ms⟶B be the projection maps of Msonto A and B respectively . Because N is stable subact of Ms, then πA(N) ⊆N and πB(N) ⊆N . Thus πA (N) ⊆A⋂N and Pb (N) ⊆B⋂N. Now, N = 1N (N) = πA (N) ⋃πB (N) ⊆ (A⋂N) ⋃(B⋂N) . The other direction of the inclusion is obvious. Hence N = (A⋂N) ⋃ (B⋂N).
In the following, we introduce a class of acts larger than the class of closed fully stable acts :
Definition (2.14): Let Msbe an S-act. A closed subact N of Msis called closed pseudo stable if f(N)⊆N for each S-monomorphism f: N ⟶Msand Msis called closed fully pseudo stable act(for simply cl-fully pseudo stable) if each subact is closed pseudo stable . The proof is essentially the same as the corresponding result in [6] , where proved that fully stable act and fully Pseudo stable are coincide :
Proposition (2.15): Every closed fully pseudo stable reversible act is closed fully stable act.■
Definition(2.16): An S-act is called terse if distinct subacts are not isomorphic.The following proposition show that the concepts of terse and closed fully pseudo stable are coincide , the proof of the following proposition by lemma(3.11) in [4]
Proposition (2.17): An S-act is cl-fully pseudo stable if and only if it is terse.

3-Acts related to cl-fully stable acts:
In the following proposition we try to put some light on therelationbetween cl-fully stable act and quasi injective, whereit gives an answer for the equation: when quasi injective acts are cl-fully stable?
Before this proposition we need the following concept.Recall that an S-act is called multiplicationif each subsystem of Msis of the form MI ,for some right ideal I of S . This is equivalent to saying that every principal subsystem is of this form[7].
In fact, since there is no relation between multiplication acts and cl-fully stable acts, so we can use it as a condition for the following proposition:
Proposition (3.1): Let Msbe multiplication S-act over commutative monoid. If Msis quasi injective, then Msis cl-fully stable act.
Proof: Let N be any closed subact of Msand f:N⟶Msbe any S-homomorphism .Since Msis multiplication, so N = IM for some ideal I of S . By quasi injectivity of Ms, f can be extended to S-homomorphism g :Ms⟶Ms. Now, f(N)=g(N)=g(IM) = Ig(M) ⊆IM=N .■Proposition(3.2): Let Msbe multiplication S-act over commutative monoid. If Msis pseudo injective, then Msis cl-fully pseudo stable act.Proof: The proof is essentially the same as the proposition (3.1) by replacing the homomorphism f: N ⟶Ms by S-monomorphism
The following proposition explain srelation between closed fully stable S-act and Baer criterion, but before we need the following concept:
Definition (3.3): Let Nsbe a subact of some act Ms . We say that N satisfiesBaer criterion, if for every S-homomorphism f:Ns⟶Ms , there exists an elements ∈S such that f(n) = ns for each n ∈Ns. An S-act Msis said to satisfy Baer criterionif every subact of Mssatisfies Baer criterion.
Proposition(3.4): If Msis closedfully stable S-act, then Mssatisfies Baer criterionfor closedsubacts (where S is a commutative monoid).
Proof: Let Nsbe a closedsubact of Msand f : Ns⟶M an S-homomorphism.Since Nsis stable, we have f(Ns) ⊆Nsand hence f(n) ∈Ns,which implies that f(n) ∈Msbut Nsis closed(this means has no proper essential extension), sothere is t ∈S such that f(n) = nt. Let w ∈Ns, hence w = nr for some r ∈S and hence f(w) ∈Ns. Sof(w) = f(nr) = f(n)r = (nt)r = n(tr) = x(nt) = (nr)t = wt. Hence there is t ∈S such that f(w) = wt for every w∈Ns. Thus Baer criterion holds for closed subacts.

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