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TYPES OF 3D SURFACE OF ROTATIONS EMBEDDED IN 4D MINKOWSKI SPACE

Anis Saad[1*], Attia Mostafa[2]

*[1] *General Engineering Department, Faculty of Engineering, Omar AlMukhtar University.*

[2]*Department of mathematics, Faculty of Arts and Science, Omar Al-Mukhtar University, Al qubba campus*.

Corresponding author:-

E-mail: *Anis.saad@omu.edu.ly*

Abstract:-

The geometry of surfaces of rotation in three dimensional Euclidean spaces has been studied widely. The rotational surfaces in three dimensional Euclidean spaces are generated byrotating an arbitrary curve about an arbitrary axis. Which should be using a type of matrices called matrices of rotation. But they are should be created by one parameter group of isometry. On the other hand, the Minkowski spaces have shorter history. In 1908 Minkowski [1864-1909] gave his talk on four dimensional real vector space, with a symmetric form of signature (+,+,+,-). In this space there are different types of vectors/ axes (space-liketime- like and null) as well as different types of curves (space-like- time-like and null). The relationship between Euclidean and Minkowskian geometry has many intriguing aspects, one of which is the manner in which formal similarity can co-exist with significantgeometric disparity. There has been considerable interest in the comparison of these twogeometries, as can be seen in the lecture notes of L'opez. In this manuscript we produce different types of surfaces of rotation in four dimensionalMinkowski spaces. And then we will provide a brief description of surfaces of rotation of 4D Minkowski spaces. Firstly consider the beginning by creating different type of matrices of rotation corresponding to the appropriate subgroup of the Lorentz group, and then generate all types of surfaces of rotation. The new work here is the spherical symmetric case which is nonabeliansubalgebra isomorphic to lie algebra. This case is known by expectation.

Keywords:-*Minkowski Spaces, surfaces of rotations, Killing vector field, Lorentz groups, Lorentz transformation, Lie group, Lie algebra*

INTRODUCTION

The geometry of surfaces of rotation in three dimensional Euclidean spaces has been studiedwidely. The rotational surfaces in three dimensional Euclidean spaces are generated byrotating an arbitrary curve about an arbitrary axis. Which should be using a type of matrices called matrices of rotation. But they are should be created by one parameter group of isometry. On the other hand, the Minkowski spaces have shorter history. In 1908 Minkowski[1864-1909] gave his talk on four dimensional real vector space, with a symmetric form of signature **(+,+,+,-)**. In this space there are different types of vectors/ axes (space-liketime- like and null) as well as different types of curves (space-like- time-like and null).

The relationship between Euclidean and Minkowskian geometry has many intriguing aspects,one of which is the manner in which formal similarity can co-exist with significantgeometric disparity. There has been considerable interest in the comparison of these twogeometries, as can be seen in the lecture notes of L'opez [2]

In this manuscript we produce different types of surfaces of rotation in four dimensionalMinkowski spaces. And then we will provide a brief description of surfaces of rotation of 4D Minkowski spaces. Firstly consider the beginning by creating different type of matrices of rotation corresponding to the appropriate subgroup of the Lorentz group, and then generate all types of surfaces of rotation. The new work here is the spherical symmetric case which is non-abeliansubalgebra isomorphic to Lie algebra. This case is known by expectation.

In section two we give a background material for two parameter subgroups of isometry, also introducing the Killing vector field which generate an equilibrium of vector spaces i.e. isometries on space. This shows the rotations and boosts in different directions, Moreover, the infinitesimal generators of null rotations. Therefore, one can see the one parameter subgroups of **SO (3,1)** representing the Lorentz transformation. And additionally seeking for generators of two parameter subgroups of **SO (3, 1)** and classified the cases of "fixing some axis" to have axis of rotation of all cases.

Section three will have types of surfaces of rotations are created by rotating an arbitrary curve (mostly time-like curve) around specific cases corresponding to the two parameter group of subgroups which fix axis. And then we have brief description of the properties of the "family of" surfaces of rotations.

And section four we have an additional case of three dimensional sub algebra, which generate the group **SO (3)** acting on two dimensional surface. This explains the spherical symmetric case. Which also generate a surface of rotation by rotating a parametric sphere on the t-axis.

Two Parameter Subgroups of Lorentz Groups.

Introduction

The matrices of rotations in $\boldsymbol{E^4}$ preserve all distances and all inner product are preserved. The analogue of a matrix of rotation in $\boldsymbol{M^{3,1}}$ with standard basis $\boldsymbol{e_x,e_y,e_z,e_t}$is denoted by $\boldsymbol{\mathcal{M}}$. The rotation matrices are replaced by Lorentz transformation such that:

$$\boldsymbol{\mathcal{M}^T\eta\mathcal{M}=\eta}.$$

Where $\boldsymbol{\eta}$ is the metric matrix of 4D Minkowski space given by:

$$\boldsymbol{\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The set of all $\boldsymbol{4}\times\boldsymbol{4}$matrices which satisfies the property above is denotedby $\boldsymbol{O(3,1)}$. If, in addition, $\mathbf{det}(\boldsymbol{\mathcal{M}})=\mathbf{1}$ and $\boldsymbol{\mathcal{M}^{4,4}}$ $\leq-\mathbf{1}$, we have the group of proper orthochronous Lorentz transformations, denoted here by $\boldsymbol{SO(3,1)}$.

Killing Vector Field

The Lorentz group is a subgroup of the diffeomorphism group of $\boldsymbol{M^{3,1}}$, and its Lie algebra can be identified with vector fields on $\boldsymbol{M^{3,1}}$. In particular, Killing vector fields arethe vectors which generate the isometries on space. We can immediately write down the general vector fields:

$$\boldsymbol{V \coloneqq \xi(x,y,z,t)\frac{\partial}{\partial x} + \eta(x,y,z,t)\frac{\partial}{\partial y} + \zeta(x,y,z,t)\frac{\partial}{\partial z} + \tau(x,y,z,t)\frac{\partial}{\partial t}}$$

Where $\boldsymbol{\xi(x,y,z,t)}\frac{\partial}{\partial x} + \boldsymbol{\eta(x,y,z,t)}\frac{\partial}{\partial y} + \boldsymbol{\zeta(x,y,z,t)}\frac{\partial}{\partial z}$and $\boldsymbol{\tau(x,y,z,t)}\frac{\partial}{\partial t}$are real functions.

We are seeking these functions $\boldsymbol{\xi(x,y,z,t)}\frac{\partial}{\partial x} + \boldsymbol{\eta(x,y,z,t)}\frac{\partial}{\partial y} + \boldsymbol{\zeta(x,y,z,t)}\frac{\partial}{\partial z}$and $\boldsymbol{(x,y,z,t)}\frac{\partial}{\partial t}$, such that, the vector field satisfies Killing vector field equation

$$\boldsymbol{gacVc;b + gcbVc;a = 0}$$

So, we have the general Killing vector fields given by:

$$V = \alpha(-y\,\partial_x + x\,\partial_y) + \beta(-z\partial_y + y\,\partial_z) + \gamma(-x\,\partial_z + z\,\partial_x)$$
$$+\,\delta\;(x\,\partial_t + t\,\partial_x) + \epsilon\,(y\,\partial_t + t\,\partial_y) + \varepsilon\,(z\,\partial_t + t\,\partial_z)$$

Where $\alpha,\beta,\gamma,\delta,\epsilon$ and $\varepsilon$are constants.

It is clearly that, we have three rotations and three boosts in different directions. But it may be helpful if we consider the infinitesimal generators of the null rotations which

$$N_x = x(\partial_t + \partial_z) + (t - z)\partial_x \;\; and \;\; N_y = y(\partial_t + \partial_z) + (t - z)\partial_y$$

Where N_x and N_y are the null rotations around the $z=t$ axis, with axes of rotation $x=0, t=z$and $y=0$,$t=z$.

One parameter subgroup of SO (3,1) representing Lorentz transformation

By this we can recognise the one parameter groups of rotations of the other generators. But we will make use of the following generators to obtain two parameter groups:

1- Two parabolic (null rotations in zt- plane) i.e. N_x and N_y.

2- Three hyperbolic (we consider only the boost of $B_z = (z\,\partial_t + t\,\partial_z)$

3- Three elliptic (we consider only the rotation around z-axis, which $R_z = (-y\,\partial_x + x\,\partial_y)$ Now, we provide the infinitesimal generatorswith given one parameter matrix group of rotation:

Type:	the infinitesimal generator	One parameter subgroup of SO(3,1) representing Lorentz transformation
Parabolic	$N_x = x(\partial_t + \partial_z) + (t - z)\partial_x$	$M_1 = \begin{pmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\alpha & 0 \\ \alpha & 0 & 1-\frac{\alpha^2}{2} & \frac{\alpha^2}{2} \\ \alpha & 0 & -\frac{\alpha^2}{2} & 1+\frac{\alpha^2}{2} \end{pmatrix}$
	$N_y = y(\partial_t + \partial_z) + (t - z)\partial_y$	$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\alpha & \alpha \\ 0 & \alpha & 1-\frac{\alpha^2}{2} & \frac{\alpha^2}{2} \\ 0 & \alpha & -\frac{\alpha^2}{2} & 1+\frac{\alpha^2}{2} \end{pmatrix}$
Hyperbolic	$B_z = (z\,\partial_t + t\,\partial_z)$	$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(\beta) & \sinh(\beta) \\ 0 & 0 & \sinh(\beta) & \cosh(\beta) \end{pmatrix}$
Elliptic	$R_z = (-y\,\partial_x + x\,\partial_y)$	$M_4 = \begin{pmatrix} \cos(\beta) & -\sin(\beta) & 0 & 0 \\ \sin(\beta) & \cos(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Generating two parameter subgroups of SO (3, 1) which are analogue of rotations in E³

The sub-algebra of the Lie algebra of the Lorentz groups can be enumerated, up to conjugancy, from which we can find the closed subgroup of the Lorentz group. See [3], chapter six for sub-algebra of the Lorentz group.

We seek two parameter group of subgroups of **SO (3, 1)** which are analogue of one parameter groups of rotation. But here we are going to find a two parameter subgroup which fix (some axis of rotation).

Then we find two dimensional sub-algebra, and hence the corresponding subgroups.

Therefore, we have three cases:

Case(1) : Two parameter group fixing the null axis located in zt-plane given by : **(0,0,1,1)** . Substitute into Killing field equation above, we find only $\{N_x, N_y\}$ is a closed sub-algebra and it is also Abelian.

So, the basis for this case is $\{N_x, N_y\}$, thus we have an Abelian subgroup of **SO(3,1)** Then N_x, N_y generate an Abelian sub-algebra consisting entirely of parabolic. So, the matrices $M_1.M_2$ will make the rotational group of matrices for this case..

Case (2): The Two parameter group fixing a space-like axis say the line given by **(0, 1, 0,0)** i.e. the yaxis. Substitute into Killing field equation above again.

Unfortunately, there are no closed sub-algebra. So there is no two dimensional sub-algebra. But if we recall that $N_x = B_x - R_y$. , we consider$\acute{N_x} = B_x + R_y$, then we see that $\{\acute{N_x}, B_z\}$ and $\{N_x, B_z\}$ each span a two dimensional sub-algebra.

So, we choose $\{N_x, B_z\}$ as a basis. And we have a nonabelian subgroup of **SO (3,1)**.

Then $\{\boldsymbol{N_x}, \boldsymbol{B_z}\}$ generate a non-abelian sub-algebra isomorephicto the Lie algebra if the affine group **A(1)**, see[4]. In this case the matrices of rotation are given by $\boldsymbol{M_1}.\boldsymbol{M_3}$or $\boldsymbol{M_3}.\boldsymbol{M_1}$, since this case is not commutative group.
The two products are different, however, for given $\boldsymbol{M_1}.\boldsymbol{M_3}$there exist $\boldsymbol{M'_1}.\boldsymbol{M'_3}$ such that $\boldsymbol{M_1}.\boldsymbol{M_3} = \boldsymbol{M'_3}.\boldsymbol{M'_1}$
Case (3): The two parameter group fixing a time-like axis is given by **(0, 0,0,1)**. Substituting the axis of rotation in the Killing vector field equation. This time again there is no two dimensional sub-algebra, see [[5] p 187].
Case (4): There is *another* two dimensional sub-algebra kown by classification [see Hall's book [3] p 163] in table 6.1 there are three groups of two dimensional sub-algebra. The first and second groups are equivalent to the cases (1) and (2). And the third one is generated by boost and rotation which is here given by $\boldsymbol{R_z}, \boldsymbol{B_z}$. Which is also an Abelian subgroup of **SO (3,1)**. So generate an abelian sub-algebra consisting of boost and rotation. So, the matrices $\boldsymbol{M_3}.\boldsymbol{M_4}$ will make the rotational group of matrices for this case. But this group does not fix any axis, and so it is not a rotation about any axis. But we can still investigate an invariant surface.
In fact, we have for this case a combination of two surfaces of rotation from$\boldsymbol{M^{2,1}}$ rotation around t axis and boost in direction of z in the plane **zt**. Now we will use the three cases above to generate special types of surfaces of rotations in Minkowski space.

Definition: *The surface* $\boldsymbol{\Sigma}$*in*$\boldsymbol{M^{3,1}}$ *is called a surface of rotation if* $\boldsymbol{\Sigma}$*is invariant by one of the three cases of two dimensional sub-group above.*
3D Surfaces of Rotations in 4D Minkowski spaces

Surfaces of rotations generated by two parabolic subgroups.
This surface is generated by entirely (null) i.e. two null rotations, without loose of generality, we take the planar curve $\boldsymbol{\gamma}$ for this surface of rotation to be the intersection of the parameterization with $\boldsymbol{x}= \boldsymbol{y}=\boldsymbol{0}$. Then assume that the curve γ lies on the zt-plane. Hence, it can be parameterized by:

$$\boldsymbol{\gamma}(\boldsymbol{w})=(\,\boldsymbol{0},\boldsymbol{0},\boldsymbol{z}(\boldsymbol{w}),\boldsymbol{t}(\boldsymbol{w})),$$

where$\boldsymbol{z}(\boldsymbol{w}),\boldsymbol{t}(\boldsymbol{w})$ are smooth functions. Also to ensure that the surface is regular, we require that, $\boldsymbol{t}(\boldsymbol{w})-\boldsymbol{z}(\boldsymbol{w})$ is positive function. Hence, the surface of rotation which will be denoted in this case by $\boldsymbol{\Sigma^1}$**,** around the line **z=t,x=y=0**. It can be parameterized by:

$$\boldsymbol{\Sigma^1}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})=\boldsymbol{M_1}(\boldsymbol{u}).\boldsymbol{M_2}(\boldsymbol{v}).\boldsymbol{\gamma}(\boldsymbol{w}),$$

So, this surface of rotation of this case is:

$$\boldsymbol{\Sigma^1}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) = \begin{pmatrix} -u\,z(w)+u\,t(w) \\ -v\,z(w)+v\,t(w) \\ (1-\frac{v^2}{2}-\frac{u^2}{2})\,z(w)+(\frac{v^2}{2}+\frac{u^2}{2})\,t(w) \\ (-\frac{v^2}{2}-\frac{u^2}{2})\,z(w)+(1+\frac{v^2}{2}+\frac{u^2}{2})\,t(w) \end{pmatrix}$$

Which has a first fundamental form of:

$$\boldsymbol{I_{\Sigma 1}} = -\boldsymbol{\partial_w} + \boldsymbol{\rho^2}(\boldsymbol{w})\boldsymbol{\partial_u} + \boldsymbol{\rho^2}(\boldsymbol{w})\boldsymbol{\partial_v}$$

Such that $\boldsymbol{z'^2}(\boldsymbol{w})-\boldsymbol{t'^2}(\boldsymbol{w})= -\boldsymbol{1}$ and $\boldsymbol{\rho}(\boldsymbol{w})= -\boldsymbol{z}(\boldsymbol{w})+\boldsymbol{t}(\boldsymbol{w})$, we may require that $\boldsymbol{\rho}(\boldsymbol{w})\neq\boldsymbol{0}$. So we can see that the first fundamental form is parameterized by one parameter variable. And it has the signature of **(-,+,+)** everywhere, which gives a Lorentz metric on it.

Surfaces of rotations generated by parabolic and boost subgroups.
This surface is generated by parabolic and boost subgroups. Again same procedure, we assume the axis of rotation is y-axis and the curve – is parametrized by

$$\boldsymbol{\gamma}(\boldsymbol{w})=(\boldsymbol{0},\boldsymbol{y}(\boldsymbol{w}),\boldsymbol{0},\boldsymbol{t}(\boldsymbol{w}))$$

Where **y(w)** and **t(w)** are smooth functions. But because this case we do not have an Abelian sub- algebra, so we have two parameterizations, we will produce both:

Surface generated by parabolic and boost:
This surface can be parametrized by:

$$\boldsymbol{\Sigma^2}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})=\boldsymbol{M_1}(\boldsymbol{u}).\boldsymbol{M_3}(\boldsymbol{v})\;\boldsymbol{\gamma}(\boldsymbol{w})$$

So, the surface of rotation is given by:

$$\boldsymbol{\Sigma^2}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) = \begin{pmatrix} (-u\sinh(v)+u\cosh(v))\,t(w) \\ y(w) \\ ((1-1/2u^2)\sinh(v)+1/2u^2\cosh(v))t(w) \\ (-1/2u^2)\sinh(v)+(1+1/2u^2)\cosh(v))\,t(w) \end{pmatrix}$$

Which has the first fundamental form of:

$$\boldsymbol{I_{\Sigma 2}} = -\boldsymbol{\partial_w} + \boldsymbol{t^2}(\boldsymbol{w})\boldsymbol{e^{-2v}}\boldsymbol{\partial_u} + \boldsymbol{t^2}(\boldsymbol{w})\boldsymbol{\partial_v}$$

This is the first fundamental form of this surface. It is clearly that it has two parameter variables. Also it does have signature of **(-, +,+)** which also give Lorentz metric on it.

Surface generated by boost and parabolic:

This surface can be parameterized by:

$$\Sigma^3(w,u,v)=M_3(\alpha).M_1(\beta)\ .\gamma(w)$$

So, the surface of rotation is given by:

$$\Sigma^1(w,u,v) = \begin{pmatrix} -u\,z(w)+u\,t(w) \\ -v\,z(w)+v\,t(w) \\ (1-\frac{v^2}{2}-\frac{u^2}{2})\,z(w)+(\frac{v^2}{2}+\frac{u^2}{2})\,t(w) \\ (-\frac{v^2}{2}-\frac{u^2}{2})\,z(w)+(1+\frac{v^2}{2}+\frac{u^2}{2})\,t(w) \end{pmatrix}$$

Which has the first fundamental form of:

$$I_{\Sigma 3} = -\partial_w + t^2(w)e^{-2v}\partial_\alpha + \beta\ t^2(w)\partial_\alpha\partial_\beta + t^2(w)(1+\beta^2)\partial_\beta$$

This is the first fundamental form of this surface. It is clearly that it has two parameter variables. Also it does have signature of **(-,+,+)** which also give Lorentz metric on it. Furthermore, it is important to note that, the coordinates of parameterization is not orthogonal. Since the first fundamental form is not diagonal in this case.

The relationship between the parameterization of Σ^2 and Σ^2

We may think that those two parameterization give the same surface of rotation but with different parameterization. So, On equating both generators of two parameter group of isometries,

□□(□). □□(□) = □□(□). □□(□)

On equating all the isomatries we have:

□ = □ , □ = □ □$^{-□}$

Or

□ = □□$^{□}$, □ = □

An explicit calculation verifies that:

□□(□). □□(□) = □□(□)□□(□ □$^{-□}$)

Surface of rotation generated by boost and rotation subgroup:

Actually, there is not two dimensional subalgebra see [[4] p 87]. But there is another two dimensional sub-algebra known by classification [see Halls book [4] p 163] in table 6.1 there are three groups of two dimensionalsub-algebra. The first and second are equivalent to cases (1) and (2) respectively. And third one is generated by boost and rotation. Which is here given by $\mathcal{Z}_2$, $\mathcal{Z}_2$. So, this surface is generated by boost and rotation, without loose of generality, we take the planar curve $\mathcal{Z}$ for this surface of rotation to be the intersection of the parameterization with $\mathcal{Z} = \mathcal{Z} = \mathcal{Z}$. Then assume that the curve γ lies on the yt-plane. Hence , it can be parameterized by:

□(□) = (□, □(□), □, □(□)),

where□(□), □(□) are smooth functions. And □(□) is positive function. The surface of rotation which will be denoted in this case by □$^{□}$,]It can be parameterized by:

□$^{□}$(□, □, □) = □□(□). □□(□). □(□),

So, the surface of rotation is given by:

$$\Sigma^4(□,□,□) = \begin{pmatrix} -\sin(v)\,y(w) \\ \cos(v)\,y(w) \\ \sinh(u)\,t(w) \\ \cosh(u)\,t(w) \end{pmatrix}$$

Now, require that □$'^{□}$(□) − □$'^{□}$(□) = −□ so the first fundamental form of

□□□ = −□□ + □$^{□}$(□)□□ + □$^{□}$(□)□□

In order to ensure that the surface is regular, we require □(□) ≠ □ ≠ □(□). This is the first fundamental form of this surface. It is clearly that it has one parameter variable. Also it does have signature of (-, +,+)which also give Lorentz metric on it. Finally, these are all types of surfaces of rotation embedded in 4D Minkowskian space. But finally we got another type of two parameter groups of isometry. This case is knowing by expectation. The idea of this is taking a parametric sphere which is also combine two parameter group of sub-algebra as we will see next.

Spherical Symmetric case

The rotations □□, □□ and □□ are three dimensional sub-algebra, they generate the group of SO (3) and SO (3) acting on a point gives a two dimensional surface. In fact, the surface of rotation is parameterized by fixing

□(□) axis and the sphere of radius □(□)in the plane (x,y,z,t(w)). This makes like rotation of parametric sphere around the time□ − □□□□.

This called the spherical symmetric case.

So, first, we know the sphere in □□parameterized by

$$□(□,□) = (□□□(□)\,□□□(□), □□□(□)\,□□□(□), □□□(□))$$

In the spherical symmetric case we therefore have the parameterization:

$$□^{□}(□,□,□) = \begin{pmatrix} \cos(u)\sin(v)\,z(w) \\ \cos(u)\cos(v)\,z(w) \\ \sin(u)\ z(w) \\ t(w) \end{pmatrix}$$

Now, if we assume that the generator is time-like. Then we can assure that $□'^{□}(□) − □'^{□}(□) = −□$ then we have:

$$□_{□□} = −□_{□} + □^{□}(□)□_{□} + □□□^{□}(□)\,□^{□}(□)□_{□}$$

This is the first fundamental form of this surface, which has the signature of **(-,+,+)** everywhere, which gives the Lorentz metric on it. Also we can observe that the first fundamental form does have two variable parameter. Then we need another parameterization of this surface. So of another conserved quantity, we choose

$$□(□,□) = (□□□(□), □□□(□)\,□□□(□), □□□(□)\,□□□(□))$$

Given by rotation around x-axix, this gives:

$$□^{□}(□,□,□) = \begin{pmatrix} \sin(\alpha)\ z(w) \\ \cos(\alpha)\ \cos(\beta)\ z(w) \\ \cos(\alpha)\ \sin(\beta)\ z(w) \\ t(w) \end{pmatrix}$$

And with the same calculation the same fundamental form is:

$$□_{□□} = −□_{□} + □^{□}(□)□_{□} + □□□^{□}(□)\,□^{□}(□)□_{□}$$

The relationship between the two parameterizations □□ and □□ given on equating the whole entries between them, on solving using matlab software, we conclude that the relationship can be given by

$$□ = □□□□□□(□□□(□)\,□□□(□)) \qquad □□□ \qquad □$$
$$= □□□□□□\left(\frac{□□□(□)}{\sqrt{□ − (□□□(□))^{□}(□□□(□))^{□}}}\right)$$

Or

$$□ = □□□□□□(□□□(□)\,□□□(□)) \qquad □□□ \qquad □$$
$$= □□□□□□\left(\frac{□□□(□)}{\sqrt{□ − (□□□(□))^{□}(□□□(□))^{□}}}\right)$$

The interesting in this surface is that, it has the signature of **(-, +,+)** everywhere, which gives the Lorentz metric on it, and it does have an orthogonal basis which orthonormal.

Conclusion and Future work

To sum up, any surface of rotation of 4D Minkowski spaces should generated by two zparameter group of isometry, this analogues the surfaces of rotation of 3D Minkowski spaces. By the beginning we are seeking a two parameter group which fix some axis of rotation. Which gotten by solving Killing vector field. Following we found three different types of two dimensional sub-Algebra. These generate two dimensional sub-groups of isometry, analogue to rotations in $\mathbb{Z}^2$. These two dimensional sub-groups of isometries are used to parameterized three different types of families surfaces of rotations embedded in 4D Minkowski space. Mostly they have an orthonormal basis on the first fundamental form. However, one of the parametrization calledΣ^2does not have orthogonal basis. Also the surfaces parametrized by Σ^2, Σ^2 are in two variables parameter in the first fundamental form.

Straight forward the case given by the parametric sphere called spherical symmetric case, parameterizations $\Sigma^2$and Σ^2 are in two variable parameter but it does have orthogonal basis, orthonormal basis, but it's not Abelian.

This work opens many researching aspects, such as studying famous 3D surfaces if they are will act on **SO(3)** to make two parameter groups of isometries, Moreover the thinking of the curves on the surfaces is valuable also. We think also for classification of all surfaces of this type/ properties. And we study other properties; such as integration over the surfaces. And CMC or minimal types.

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