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CORRELATION FUNCTIONS FOR LOTKA-VOLTERRA MODEL

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Abstract:-

This work is about studying biological system interactions which founded in two types. Also it is one of modeling topics that based on the use of non-linear ordinary differential equations. Consequently, the mutual affect between the interactive groups is estimated. In addition the elements of the same group to chronological correlation functions are of second order form. Therefore, the reflection of mutual affects is due to the existence of the set of solutions. First we choose a simple model of single species of biological system, where we can get the solutions of the governing equations. And then calculate the correlation functions related to the solutions. After that we study the evolution of the lotka-volterra interacting model, then we get the solutions of nonlinear system by approximated method, and evaluate the correlation functions.

Keywords:-*Non-linear (ODEs), Single species, Biological systems, correlation functions, Lotka-volterra model.*

INTRODUCTION

As Kapur [3] says in winding and deepening the scope of mathematical biosciences, we are not first restricted to the use of mathematical techniques already known. In fact, one role of mathematicians interested in biological and medical problems is to evolve new mathematical methods for dealing with the complex situations. Kapur [4] analyzed the stability of continuous and discrete population models, in his work, he examined the stability of models for interacting species with same points of equilibrium, using difference equation system. In [5, 6] it were studied moments for some general birth death processes, determined the moments of all orders for the probability distribution of ultimate size of the population the generalized birth and death processes with twin birthsusingthe difference equation technique.It had been shown in [7] that equilibrium point for predator-prey models with discrete time lage is always unstable.The use of hypergeometric functions to generalized Birth and death processes is given by Kapur in [8] he enumerates the number of possible relations between probabilitiesof ultimate extinction of birth and death processes.In another paper [9] titled some mathematical models for population growth, itdecided one of the most successful models for explainingthe growth of populations of bacteria and even of humans is the so-called Logistic model all orders for the probability distribution of ultimate size of the population the generalized birth and death processes with twin births using the difference equation technique. It had been shown in [7] that equilibrium point for predator-prey models with discrete time lage is always unstable. The use of hypergeometric functions to generalized Birth and death processes is given by Kapur in [8] he enumerates the number of possible relations between probabilities of ultimate extinction of birth and death processes. In another paper [9] titled some mathematical models for population growth, it decided one of the most successful models for explaining the growth of populations of bacteria and even of humans is the so-called Logistic model.

The combined use of game theory and modified volterra equations in describing the population dynamics is treated by Kapur [10] he concluded that volterr's system of differential equations for n interacting species has been modified and it is shown that the modified system is equivalent to the system of cubic differential equations obtained earlier for animal conflicts from considerations of theory of games. In his study of nonlinear continuous- time discrete- age- scale population models, he showed that whenever the corresponding linear model predicts exponential growth, the nonlinear model gives a stable equilibrium age-distribution [11]. The study of the effect of harvesting on competing population. he showed that for the simplest competition model for two species there are four nondegenerate possibilities for ultimate behavior according as (i) the first species alone survives, (ii) the second species alone survives, (iii) the two species coexist, in stable equilibrium and (iv) the two species coexist in unstable equilibrium, and the survival of first or second species depends on the initial population size [12].

The optimal foraging and predator-prey dynamics by Vlastmil K-Řivan in [13] is treated for a system consisting of population of predators and two types of prey. The dynamics is described by differential equations with controls the choice of these controls is based on the standard assumption in the theory of optimal foraging which requires that each predator maximizes the net rate of energy intake during foraging. The proposed a model to describe the interaction between a diseased fish population and their predators he analyzed the stability of equilibrium points for a large range of parameter values, it established the existence and uniqueness of solutions and found that the solutions are uniformly bounded for all non-negative initial conditions [14]. In model which is proposed by chattopadhyay and Bairagi predicts that a deadly disease and predator population cannot co-exist [14]. In [15] manipulated the spatial dynamics and cross-correlation in a transient predator-prey system. It was found that during the exponential population growth, beetles were generally strongly negatively cross-correlated with the prey at local spatial scales. It simulates the partiallyextended interactions in predator-prey coupled map lattice model and used this model in investigating the effects of global and local prey reproduction, in the presence and absence of global stochasticity, on predator and prey spatial structuring and crosscorrelation.

The effects of correlated interactions in a biological convolution model with individual-based dynamics is a research given by [16]. In his study it recognized, models of biological convolution in which a species is defined by a genome in the form of a finite bitstring, and the interaction between species i and j are given by a fiseed matrix with independent, randomly distributed elements M_{ij}. This means that species whose genotypes. Differ even by a single bit may have completely different phenotypes, as defined by their interactions with the other species. [17] Introduces the concept of coherence into the predation of biological system through his work titled on the influence of noise on the critical and oscillatory behavior of a predatorprey models: coherent stochastic resonance at the proper frequency of the system. In [18] introduces variational iteration method for solving multispecies lotka-volterra equations. And [19] studied a modified algorithm for approximate solutions of lotkavolterra systems. While [20] has proposed utility functions and lotka-volterra model: A possible connection in predator-prey game. Susmita paul [21] introduces numerical solution of lotka-volterra prey predator model by using runge-kutta fehlberg method and laplace a domain decomposition method. Yigit Aksoy in [22] he discussed application of perturbation-iteration method to lotka-volterra equations. Noura taher in [23] studied analysis of hybrid dynamical systems with an application in biological systems.

This paper is about studying biological system interactions which founded in two types. First we choose a simple model of single species of biological system, where we can get the solutions of the governing equations. And then calculate the correlation functions related to the solutions. After that we study the evolution of the lotka-volterra interacting model, then we get the solutions of non-linear system by approximated method, and evaluate the correlation functions.

LOGISTIC MODELS

The logistic models are based on a Logistic equation defined as follows:

Let *x (t)* be the population at time *t; b* and *d* be the intrinsic or specific birth and death rates, respectively. This leads to Malthus model (1798) [3]:

$$dx/dt = bx - dx = (b-d)x = ax. \quad \mathbf{(2.1)}$$

when b , d and a are constants, by integration we get:

$$x(t) = x(0)e^{at} \quad \mathbf{(2.2)}$$

so that the population grows exponentially if a >0, decays exponentially if a <0, and remains constant if a=0. In general, b is a monotonic decreasing function of x and d is a monotonic increasing function of x so that a is a monotonic decreasing function of x.

$$\text{hence we write: } dx/dt = x\lfloor b(x) - d(x) \rfloor = xa(x), \quad \dot{b}(x) < 0, \ \dot{d}(x) > 0, \quad \mathbf{(2.3)}$$

The simplest case arises when

$$b(x) = b_1 - b_2 x \ , \ d(x) = d_1 + d_2 x, \ a(x) = a - cx, \ a = b_1 - d_1, \ c = b_2 + d_2, \quad b_1, b_2, d_1, d_2, a, c > 0, \quad \mathbf{(2.4)}$$

where we assume for the present that: $x \leq b_1 / b_2$. **(2.5)**

however, $x > b_1/ b_2$, the birth rate is taken as zero. from (2.3) and (2.4), we get Logistic model

$$\frac{dx}{dt} = x(a - cx) \quad \mathbf{(2.6)}$$

Integrating (2.6) we obtain

$$x(t) = \frac{a/c}{1 + \left(\frac{a/c}{x(0)} - 1\right)e^{-at}} \quad \mathbf{(2.7)}$$

so that, as $t \to \infty$, $x(t) \to a/c$. If $x(0) < a/c$, then dx/dt is always positive and $x(t)$ Increases to a limiting population size a/c. If $x(0) > a/c$, then dx/dt is always negativeand $x(t)$ decreases to a/c (see Fig. 2.1). The final population size in any case is a/c, and since

$$\frac{a}{c} = \frac{b_1 - d_1}{b_2 + d_2} < \frac{b_1}{b_2} \quad \mathbf{(2.8)}$$

when $x(0) < a/c$, condition (2.5) is always satisfied and birth rate always remains positive, Thus we shall assume that $x(0) < a/c$.

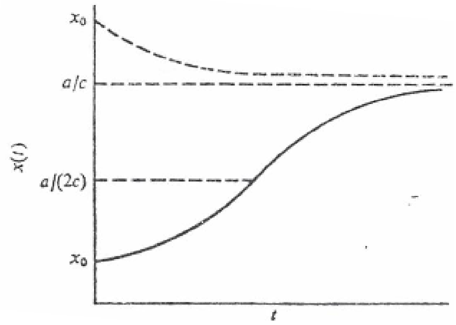

Fig. 2.1Logistic curve

Differentiating (2.6), we obtain:

$$\frac{d^2x}{dt^2} = a - 2cx = 2c\left(\frac{a}{2c} - x\right). \quad \mathbf{(2.9)}$$

If $x(0) < a/(2c)$, then dx/dt increases as x varies from $x(0)$ to $a/(2c)$ and decreases as x varies from $a/(2c)$ to a/c . dx/dt Changes from an increasing to a decreasing function at $x = a/(2c)$, and d^2x/dt^2 vanishes when $x = a/(2c)$ so that there is a point of inflection in the population growth curve when half the final population size is reached. From (2.7), the point of inflection occurs at time:

$$t_1 = \frac{1}{a}\left\{\ln\left(\frac{a/c}{x(0)} - 1\right)\right\}. \quad \mathbf{(2.10)}$$

If x (0)> a/ (2c), there is no point of inflection.

Generalized Logistic Models

This model is a member of class of models which gives sigmoid growth cure with a limiting population size and a point of inflection. In model:

$$\frac{dx}{dt} = xa(x) \quad , \dot{a}(x) < 0 \qquad (2.1.1)$$

$a(x)$ is any positive monotonic decreasing function of x. If $a(x)$ vanishes at $x = k$, then k gives limiting size of the population.\

Since: $$\frac{d^2x}{dt^2} \equiv \phi(x) = x\dot{a}(x) + a(x), \qquad (2.1.2)$$

$a(x)$ vanishes at $x = k$, and $\dot{a}(x)$ is always negative, we have:

$$\phi(k) = k\dot{a}(k) < 0 \qquad (2.1.3)$$

so that, at $x = k$, dx/dt is zero, d^2x/dt^2 is negative, and x attains its maximum value, also

$$\phi(x_0) = x_0\dot{a}(x_0) + a(x_0), \quad x_0 = x(0) \qquad (2.1.4)$$

If $\phi(x_0) > 0$, then $\phi(x)$ will, n general, be a continuous function which is positive at x_0 and negative at k and, therefore, it will at vanish some point between x_0 and, k. Hence a necessary condition for the existence of a point of inflection is that.

$$x_0\dot{a}(x_0) + a(x_0) > 0. \qquad (2.1.5)$$

Thus, when $a(x)$ is a positive monotonic decreasing function of x, (2.3) gives a Generalized logistic model with a limiting size k and a point of inflection at $k_1 < k$ if (2.1.5) is satisfied and:

$$a(k) = 0, \; k_1\dot{a}(k_1) + a(k_1) = 0, \; k_1 < k. \qquad (2.1.6)$$

Another generalization of the logistic model of some interest is given by:

$$\frac{dx}{dt} = \frac{a}{\alpha} x \left[1 - \left(\frac{x}{k}\right)^{\alpha}\right] \qquad (2.1.7)$$

when $\alpha = 1$, (2.1.7) reduces to the logistic model ; when $\alpha \to 0$, it gives:

$$\frac{dx}{dt} = ax \ln \frac{k}{x}. \qquad (2.1.8)$$

or $$dy/dt = a(\ln k - y), \; y = \ln x \qquad (2.1.9)$$

so that ln x satisfies a linear differential equation. Integrating (2.1.8) or (2.1.9), we obtain the Gompertz growth law

$$x = x_0 \exp\left[(\ln \frac{k}{x_0})(1 - e^{-at})\right]. \qquad (2.1.10)$$

Evolution Equations and Solutions of the Generalized Logistic Model

In the generalized logistic model we have:

$$\frac{dx}{dt} = \frac{a}{\alpha} x \left[1 - \left(\frac{x}{k}\right)^{\alpha}\right], \; \alpha \text{ is a real number} \qquad (2.2.1)$$

Integrating equation (2.2.1) we have:

$$x(t) = \frac{k^{\alpha}}{(x(0))^{\alpha-1}\left(1 + \left[\left(\frac{k}{x(0)}\right)^{\alpha} - 1\right] e^{-\frac{a}{\alpha}t}\right)} \qquad (2.2.2)$$

For $\alpha = 1$ and $k = \frac{a}{c}$ we have the Pearl-Verhulst Logistic Model:

$$\frac{dx}{dt} = ax - cx^2 \qquad (2.2.3)$$

Correlation Functions

we shall select the generalized logistic model to compute the pair-correlation function associated with its solution, then evaluate that function for Pearl-Verhulst model as a special case when $\alpha = 1, \; k = \frac{a}{c}$.

For the generalized logistic model from equation (2.2.2) we have:

$$\langle x(t)\rangle = \frac{\alpha k^{\alpha}}{a(x(0))^{\alpha-1}T}[\ln\left(\frac{1+c_1 e^{-\frac{a}{\alpha}T}}{1+c_1}\right)+\frac{a}{\alpha}T] \quad \textbf{(2.3.1)}$$

and: $$\langle x(t)^2\rangle = \frac{\alpha k^{2\alpha}}{a(x(0))^{2(\alpha-1)}T}[\frac{a}{\alpha}T+\ln\left(\frac{1+c_1 e^{-\frac{a}{\alpha}T}}{1+c_1}\right)-\frac{c_1(1-e^{\frac{a}{\alpha}T})}{(1+c_1)(1+c_1 e^{\frac{a}{\alpha}T})}] \quad \textbf{(2.3.2)}$$

From equations (2.3.1), (2.3.2) and by definition in [1, 2] we can calculate $G^{(2)}(t)$.

Iterative Solution of Lotka-Volterra Model for Two Specie Systems

In Lotka-Volterra model the basic equations can be written as:

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_1 N_1 N_2,\; a_1, \alpha_1 > 0, \quad \textbf{(3.1)}$$

$$\frac{dN_2}{dt} = -a_2 N_2 + \alpha_2 N_1 N_2,\; a_2, \alpha_2 > 0$$

(3.2)

Let $N_1(t) = N_{10} + \varepsilon_1 \eta_1(t)$ and $N_2(t) = N_{20} + \varepsilon_2 \eta_2(t)$.

Substituting (3.2) into (3.1) then neglect the terms containing powers of ε greater than or equal to 2 we have

$$\frac{d\eta_1}{dt} = \beta_1 \eta_1(t) + \gamma_1 \eta_2(t) + \omega_1, \quad \textbf{(3.3)}$$

$$\frac{d\eta_2}{dt} = \beta_2 \eta_1(t) + \gamma_2 \eta_2(t) + \omega_2.$$

Where: $\beta_1 = a_1 - \alpha_1 N_{20}$, $\gamma_1 = \frac{-\alpha_1 N_{10}\varepsilon_2}{\varepsilon_1}$, $\omega_1 = \frac{a_1 N_{10} - \alpha_1 N_{120}}{\varepsilon_1}$, $\beta_2 = \frac{\alpha_2 N_{20}\varepsilon_1}{\varepsilon_2}$,

$\gamma_2 = -a_2 + \alpha_2 N_{10}$, $\omega_2 = \frac{-a_2 N_{20} + \alpha_2 N_{120}}{\varepsilon_2}$, and $N_{120} = N_{10} N_{20}$.

In matrix form equation (3.3) take the form

$$\frac{d}{dt}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{pmatrix}\beta_1 & \gamma_1\\ \beta_2 & \gamma_2\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} + \begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \quad \textbf{(3.4)}$$

We shall solve the equation

$$\frac{d}{dt}\begin{pmatrix}\eta_1+\Omega_1\\ \eta_2+\Omega_2\end{pmatrix} = A\begin{pmatrix}\eta_1+\Omega_1\\ \eta_2+\Omega_2\end{pmatrix} \quad \textbf{(3.5)}$$

where: $A = \begin{pmatrix}\beta_1 & \gamma_1\\ \beta_2 & \gamma_2\end{pmatrix}$ and $\begin{pmatrix}\Omega_1\\ \Omega_2\end{pmatrix} = A^{-1}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$

Assuming that:

$$(\eta + \Omega)(t) = \varepsilon e^{\lambda t} \quad \textbf{(3.5a)}$$

Leads to the set of linear algebraic equations:

$$\begin{pmatrix}\beta_1 - \lambda & \gamma_1\\ \beta_2 & \gamma_2 - \lambda\end{pmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{12}\end{pmatrix} = 0 \quad \textbf{(3.6)}$$

Which determine the eigenvalues and eigenvector of A

Now the eigen values of A are given by: $|A - \lambda I| = 0$ **(3.7)**

Or $$\begin{vmatrix}\beta_1 - \lambda & \gamma_1\\ \beta_2 & \gamma_2 - \lambda\end{vmatrix} = 0 \quad \textbf{(3.8)}$$

Then: $\lambda^2 - b\lambda + c = 0$

Where: $b = \beta_1 + \gamma_2$ and $c = \beta_1\gamma_2 - \beta_2\gamma_1$ **(3.8a)**

The roots are:

$$\lambda_{1,2} = \frac{b \pm \sqrt{b^2 - 4c}}{2} \quad (3.8b)$$

For $b^2 > 4c \Longrightarrow \lambda_1 \lambda_2$ are real number,

For $b^2 = 4c \Longrightarrow \lambda_1 = \lambda_2 = \frac{b}{2}$ are equal real number,

For $b^2 < 4c \Longrightarrow \lambda_1, \lambda_2 = \overline{\lambda_1}$ are complex number.

Case (1): If $b^2 > 4c$ we obtain $\lambda_1 = \lambda_{11}$ and $\lambda_2 = \lambda_{12}$ **(3.9)**

Where: $\lambda_{11} = \frac{b + \sqrt{b^2 - 4c}}{2}$ and $\lambda_{12} = \frac{b - \sqrt{b^2 - 4c}}{2}$

Form (3.8b) into (3.6) can be obtain the eigenvectors

$$\varepsilon^{(1)} = \begin{pmatrix} 1 \\ a_{12} \end{pmatrix} \text{ and } \varepsilon^{(2)} = \begin{pmatrix} 1 \\ a_{22} \end{pmatrix} \quad (3.9a)$$

Where: $a_{12} = \frac{-(\beta_1 - \lambda_{11})}{\gamma_1}$ and $a_{22} = \frac{-(\beta_1 - \lambda_{12})}{\gamma_1}$

Substituting Form (3.9a) into (3.5a) we have:

$$(\eta + \Omega)^{(1)}(t) = \begin{pmatrix} 1 \\ a_{12} \end{pmatrix} e^{\lambda_{11} t} \text{ and } (\eta + \Omega)^{(2)}(t) = \begin{pmatrix} 1 \\ a_{22} \end{pmatrix} e^{\lambda_{12} t} \quad (3.9b)$$

The general solution of the equation (3.5) are:

$$\eta_1(t) = c_1 e^{\lambda_{11} t} + c_2 e^{\lambda_{12} t} - \Omega_1,$$

$$\eta_2(t) = c_1 a_{12} e^{\lambda_{11} t} + c_2 a_{22} e^{\lambda_{12} t} - \Omega_2. \quad (3.9c)$$

Form (3.9c) into (3.2) we have:

$$N_1(t) = d_1 e^{\lambda_{11} t} + d_2 e^{\lambda_{12} t} + d_3,$$

$$N_2(t) = D_1 e^{\lambda_{11} t} + D_2 e^{\lambda_{12} t} + D_3. \quad (3.10)$$

Where: $d_1 = c_1 \varepsilon_1$, $d_2 = c_2 \varepsilon_1$, $d_3 = N_{10} - \varepsilon_1 \Omega_1$, $D_1 = c_1 a_{12} \varepsilon_2$, $D_2 = c_2 a_{22} \varepsilon_2$

and $D_3 = N_{20} - \varepsilon_2 \Omega_2$.

Case (2): If $b^2 = 4c$ we obtain: $\lambda_1 = \lambda_2 = \lambda_{13}$ **(3.11)**

Where: $\lambda_{13} = \frac{b}{2}$

Form (3.11) into (3.6) can be obtain the eigenvectors

$$\varepsilon^{(1)} = \varepsilon^{(2)} = \begin{pmatrix} 1 \\ a_{13} \end{pmatrix} \quad (3.11a)$$

Where: $a_{13} = \frac{-(\beta_1 - \lambda_{13})}{\gamma_1}$.

Substituting Form (3.11a) into (3.5a) we have

$$(\eta + \Omega)^{(1)}(t) = \begin{pmatrix} 1 \\ a_{13} \end{pmatrix} e^{\lambda_{13} t},$$

$$(\eta + \Omega)^{(2)}(t) = \left[\begin{pmatrix} 1 \\ a_{13} \end{pmatrix} t + \begin{pmatrix} a_{23} \\ a_{33} \end{pmatrix} \right] e^{\lambda_{13} t} \quad (3.11b)$$

Where: $a_{23} = \frac{\gamma_2 \lambda_{13} - a_{13} \gamma_1}{(\beta_1 - \lambda_{13})(\gamma_2 - \lambda_{13}) - \gamma_1 \beta_2}$ and $a_{33} = \frac{1 - (\beta_1 - \lambda_{13}) a_{23}}{\gamma_1}$

$$\eta_1(t) = c_1' e^{\lambda_{13} t} + c_2 t e^{\lambda_{13} t} - \Omega_1, \quad (3.11c)$$

$$\eta_2(t) = C_1' e^{\lambda_{13}t} + c_2 a_{13} t e^{\lambda_{13}t} - \Omega_2.$$

Where: $c_1' = c_1 + c_2 a_{23}$ and $C_1' = c_1 a_{13} + c_2 a_{33}$.

Form (3.11c) into (3.2) we have:

$$N_1(t) = d_1' e^{\lambda_{13}t} + d_2 e^{\lambda_{13}t} + d_3,$$
$$N_2(t) = D_{12} e^{\lambda_{13}t} + D_{22} t e^{\lambda_{13}t} + D_3. \quad \textbf{(3.12)}$$

Where: $d_1' = c_1' \varepsilon_1$, $D_{12} = C_1' \varepsilon_2$ and $D_{22} = c_2 a_{13} \varepsilon_2$.

Case (3): If $b^2 < 4c$ we obtain: $\lambda_1 = \lambda_{13} + i\lambda_{23}$ and $\lambda_2 = \lambda_{13} - i\lambda_{23}$ **(3.13)**

Where: $\lambda_{23} = \dfrac{\sqrt{4c - b^2}}{2}$.

Form (3.13) into (3.6) can be obtain the eigenvectors:

$$\varepsilon^{(1)} = \begin{pmatrix} 1 \\ a_{14} - ib_{14} \end{pmatrix} \text{ and } \varepsilon^{(2)} = \begin{pmatrix} 1 \\ a_{14} + ib_{14} \end{pmatrix} \quad \textbf{(3.13a)}$$

Where: $a_{14} = \dfrac{-\beta_1 + \lambda_{13}}{\gamma_1}$, $b_{14} = \dfrac{\lambda_{23}}{\gamma_1}$

Substituting Form (3.13a) into (3.5a) we have

$$(\eta + \Omega)^{(1)}(t) = \begin{pmatrix} 1 \\ a_{14} - ib_{14} \end{pmatrix} e^{\lambda_{13} + i\lambda_{23}},$$
$$(\eta + \Omega)^{(2)}(t) = \begin{pmatrix} 1 \\ a_{14} + ib_{14} \end{pmatrix} e^{\lambda_{13} - i\lambda_{23}}. \quad \textbf{(3.13b)}$$

The general solution of the equation (3.5) are:

$$\eta_1(t) = c_1 \cos\lambda_{23} t e^{\lambda_{13}t} + c_2 \sin\lambda_{23} t e^{\lambda_{13}t} - \Omega_1,$$
$$\eta_2(t) = (c_1 a_{14} - c_2 b_{14})\cos\lambda_{23} t e^{\lambda_{13}t} + (c_1 b_{14} + c_2 a_{14})\sin\lambda_{23} t e^{\lambda_{13}t} - \Omega_2 \quad \textbf{(3.13c)}$$

Into (3.2) we have:

$$N_1(t) = d_1 \cos\lambda_{23} t e^{\lambda_{13}t} + d_2 \sin\lambda_{23} t e^{\lambda_{13}t} + d_3,$$
$$N_2(t) = D_{13} \cos\lambda_{23} t e^{\lambda_{13}t} + D_{23} \sin\lambda_{23} t e^{\lambda_{13}t} + D_3. \quad \textbf{(3.14)}$$

Where: $D_{13} = (c_1 a_{14} - c_2 b_{14})\varepsilon_2$ and $D_{23} = (c_1 b_{14} + c_2 a_{14})\varepsilon_2$.

We can treat the nonlinear problem iteratively considering $N_1(t)$, $N_2(t)$ from three cases as a 1st approximation $(N_1^{(1)}(t), N_2^{(1)}(t))$ then the 2nd iterative solution can be obtained from

$$\frac{dN_1^{(2)}}{dt} = a_1 N_1^{(1)} - \alpha_1 N_1^{(1)} N_2^{(1)},$$
$$\frac{dN_1^{(2)}}{dt} = -a_2 N_2^{(1)} + \alpha_2 N_1^{(1)} N_2^{(1)}. \quad \textbf{(3.15)}$$

Form equations (3.10) or (3.12) or (3.14) into (3.15) and integration $(N_1^{(2)}(t), N_2^{(2)}(t))$.
Similarly can be obtained the 3rd approximation $(N_1^{(3)}(t), N_2^{(3)}(t))$.

Second-Order Correlation Functions

To evaluate the pair correlation function between $N_1(t)$ and $N_2(t)$ we need $\langle N_1(t)\rangle$, $\langle N_2(t)\rangle$ and $\langle N_1(t) N_2(t)\rangle$ using equation in [1,2] since $N_1(t)$ and $N_2(t)$ are given in Sec 3 we shall evaluate each expectation value of *f(t)*

In **case (1)**:From equation (3.10) we have:

$$\langle N_1(t)\rangle = x_1 + \frac{1}{T}[x_2 e^{\lambda_{11}T} + x_3 e^{\lambda_{12}T} - x_4] \quad \textbf{(4.1)}$$

where: $x_1 = d_3$, $x_2 = \frac{d_1}{\lambda_{11}}$, $x_3 = \frac{d_2}{\lambda_{12}}$ and $x_4 = x_2 + x_3$.

$$\langle N_2(t)\rangle = y_1 + \frac{1}{T}[y_2 e^{\lambda_{11}T} + y_3 e^{\lambda_{12}T} - y_4] \quad \textbf{(4.2)}$$

where: $y_1 = D_3$, $x_2 = \frac{D_1}{\lambda_{11}}$, $x_3 = \frac{D_2}{\lambda_{12}}$ and $y_4 = y_2 + y_3$.

$$\langle [N_1(t)]^2\rangle = x_{11} + \frac{1}{T}[x_{12} e^{\lambda_{11}T} + x_{13} e^{\lambda_{12}T} + x_{14} e^{2\lambda_{11}T} +$$

(4.3)

$$x_{15} e^{2\lambda_{12}T} + x_{16} e^{(\lambda_{11}+\lambda_{12})T} - x_{17}]$$

where: $x_{11} = d_3^2$, $x_{12} = \frac{2d_1 d_3}{\lambda_{11}}$, $x_{13} = \frac{2d_1 d_3}{\lambda_{12}}$, $x_{14} = \frac{d_1^2}{2\lambda_{11}}$, $x_{15} = \frac{d_2^2}{2\lambda_{12}}$, $x_{16} = \frac{2d_1 d_3}{(\lambda_{11}+\lambda_{11})}$ and $x_{17} = x_{12} + x_{13} + x_{14} + x_{15} + x_{16}$.

$$\langle [N_2(t)]^2\rangle = y_{11} + \frac{1}{T}[y_{12} e^{\lambda_{11}T} + y_{13} e^{\lambda_{12}T} + y_{14} e^{2\lambda_{11}T} +$$

$$y_{15} e^{2\lambda_{12}T} + y_{16} e^{(\lambda_{11}+\lambda_{12})T} - y_{17}] \quad \textbf{(4.4)}$$

where $y_1 = D_3^2$, $y_{12} = \frac{2D_1 D_3}{\lambda_{11}}$, $y_{13} = \frac{2D_1 D_3}{\lambda_{12}}$, $y_{14} = \frac{D_1^2}{2\lambda_{11}}$, $y_{15} = \frac{D_2^2}{2\lambda_{12}}$, $y_{16} = \frac{2D_1 D_3}{(\lambda_{11}+\lambda_{12})}$ and $y_{17} = y_{12} + y_{13} + y_{14} + y_{15} + y_{16}$.

$$\langle N_1(t), N_2(t)\rangle = E_1 + \frac{1}{T}[E_2 e^{\lambda_{11}T} + E_3 e^{\lambda_{12}T} + E_4 e^{(\lambda_{11}+\lambda_{12})T} +$$

$$E_5 e^{2\lambda_{11}T} + E_6 e^{2\lambda_{12}T} - E_7] \quad \textbf{(4.5)}$$

where: $E_1 = d_3 D_3$, $E_2 = \frac{d_1 D_3 + d_3 D_1}{\lambda_{11}}$, $E_3 = \frac{d_2 D_3 + d_3 D_2}{\lambda_{12}}$, $E_4 = \frac{d_1 D_2 + d_2 D_1}{\lambda_{11}+\lambda_{12}}$, $E_5 = \frac{d_1 D_1}{2\lambda_{11}}$, $E_6 = \frac{d_2 D_2}{2\lambda_{12}}$ and $E_7 = E_2 + E_3 + E_4 + E_5 + E_6$.

In case (2): From equation (3.12) we have

$$\langle N_1(t)\rangle = z_1 + \frac{1}{T}[z_2 e^{\lambda_{13}T} + z_3 T e^{\lambda_{13}T} - z_2] \quad \textbf{(4.6)}$$

where: $z_1 = d_3$, $z_2 = \frac{d_1^1}{\lambda_{13}} - \frac{d_2}{\lambda_{13}^2}$ and $z_3 = \frac{d_2}{\lambda_{13}}$.

$$\langle N_2(t)\rangle = w_1 + \frac{1}{T}[w_2 e^{\lambda_{13}T} + w_3 T e^{\lambda_{13}T} - w_2] \quad \textbf{(4.7)}$$

where: $w_1 = D_3$, $w_2 = \frac{D_{12}}{\lambda_{13}} - \frac{D_{22}}{\lambda_{13}^2}$ and $w_3 = \frac{D_{22}}{\lambda_{13}}$.

$$\langle [N_1(t)]^2\rangle = z_{11} + \frac{1}{T}[z_{12}e^{\lambda_{13}T} + z_{13}e^{2\lambda_{13}T} + z_{14}Te^{\lambda_{13}T} + z_{15}Te^{2\lambda_{13}T} + z_{16}T^2e^{2\lambda_{13}T} - z_{17}]. \quad \textbf{(4.8)}$$

where: $z_{11} = d_3^2$, $z_{12} = (\frac{2d_1^1 d_3}{\lambda_{13}} - \frac{2d_2 d_3}{\lambda_{13}^2})$, $z_{13} = (\frac{d_1^1}{2\lambda_{13}} - \frac{d_1^1 d_2}{2\lambda_{13}^2} + \frac{d_2}{4\lambda_{13}^3})$,

$z_{14} = \frac{2d_2 d_3}{\lambda_{13}}$, $z_{15} = (\frac{d_1^1 d_2}{\lambda_{13}} - \frac{d_2^2}{2\lambda_{13}^2})$, $z_{16} = \frac{d_2^2}{2\lambda_{13}}$ and $z_{17} = z_{12} + z_{13}$.

$$\langle [N_2(t)]^2\rangle = w_{11} + \frac{1}{T}[w_{12}e^{\lambda_{13}T} + w_{13}e^{2\lambda_{13}T} w_{14}Te^{\lambda_{13}T} + w_{15}Te^{2\lambda_{13}T} + w_{16}T^2e^{2\lambda_{13}T} - w_{17}] \quad \textbf{(4.9)}$$

where: $w_{11} = D_3^2$, $w_{12} = (\frac{2D_{12}D_3}{\lambda_{13}} - \frac{2D_{22}D_3}{\lambda_{13}^2})$, $w_{13} = (\frac{D_1^2}{2\lambda_{13}} - \frac{D_{12}D_{22}}{2\lambda_{13}^2} + \frac{D_2^2}{4\lambda_{13}^3})$,

$w_{14} = \frac{2D_2 D_3}{\lambda_{13}}$, $w_{15} = (\frac{D_{12}D_{22}}{\lambda_{13}} - \frac{D_2^2}{2\lambda_{13}^2})$, $w_{16} = \frac{D_{22}^2}{2\lambda_{13}}$ and $w_{17} = w_{12} + w_{13}$.

$$\langle N_1(t)N_2(t)\rangle = z_{21} + \frac{1}{T}[z_{22}e^{\lambda_{13}T} + z_{23}e^{2\lambda_{13}T} + z_{24}Te^{\lambda_{13}T} +$$

$$z_{25}Te^{\lambda_{13}T} + z_{26}T^2e^{2\lambda_{13}T} - z_{27}]. \quad \textbf{(4.10)}$$

where: $z_{21} = F_1$, $z_{22} = (\frac{F_2}{\lambda_{13}} - \frac{D_4}{\lambda_{13}^2})$, $z_{23} = (\frac{F_3}{2\lambda_{13}} - \frac{F_5}{4\lambda_{13}^2} + \frac{F_6}{4\lambda_{13}^3})$,

$z_{24} = \frac{F_4}{\lambda_{13}}$, $z_{25} = (\frac{F_5}{2\lambda_{13}} - \frac{F_6}{2\lambda_{13}^2})$, $F_{26} = \frac{F_6}{2\lambda_{13}}$ and $z_{27} = z_{22} + z_{23}$.

Since: $F_1 = d_3 D_3$, $F_2 = d_1' D_3 + d_3 D_{12}$, $F_3 = d_1' D_{13}$, $F_4 = d_2 D_3 + d_3 D_{22}$,

In case (3): From equation (3.14) we have:

$$\langle N_1(t)\rangle = P_1 + \frac{1}{T}[P_2 \cos\lambda_{23}Te^{\lambda_{13}T} + P_3 \sin\lambda_{23}Te^{\lambda_{13}T} - P_1]. \quad \textbf{(4.11)}$$

where: $P_1 = d_3$, $P_2 = \frac{d_1\lambda_{13} + d_2\lambda_{23}}{\lambda_{13}^2 + \lambda_{23}^2}$ and $P_3 = \frac{d_1\lambda_{23} + d_2\lambda_{13}}{\lambda_{13}^2 + \lambda_{23}^2}$.

$$\langle N_2(t)\rangle = Q_1 + \frac{1}{T}[Q_2 \cos\lambda_{23}Te^{\lambda_{13}T} + Q_3 \sin\lambda_{23}Te^{\lambda_{13}T} - Q_1] \quad \textbf{(4.12)}$$

where: $Q_1 = D_3$, $Q_2 = \frac{D_{13}\lambda_{13} - D_{23}\lambda_{23}}{\lambda_{13}^2 + \lambda_{23}^2}$, and $Q_3 = \frac{D_{13}\lambda_{23} + D_{23}\lambda_{13}}{\lambda_{13}^2 + \lambda_{23}^2}$.

$$\langle [N_1(t)]^2 \rangle = P'_{11} + \frac{1}{T}[P'_{12}e^{2\lambda_{13}T} + P'_{13}\cos\lambda_{23}Te^{2\lambda_{13}T} + P'_{14}\sin\lambda_{23}Te^{\lambda_{13}T} +$$
$$P'_{15}\cos 2\lambda_{23}Te^{2\lambda_{13}T} + P'_{16}\sin\lambda_{23}T^2e^{2\lambda_{13}T} - P'_{17}] \quad \textbf{(4.13)}$$

where: $P'_{11} = P_{11}$, $P'_{12} = \frac{P_{12}}{2\lambda_{13}}$, $P'_{13} = \frac{P_{13}\lambda_{13} - P_{14}\lambda_{23}}{\lambda_{13}^2 + \lambda_{23}^2}$, $P'_{14} = \frac{P_{13}\lambda_{23} + P_{14}\lambda_{13}}{\lambda_{13}^2 + \lambda_{23}^2}$,

$P'_{15} = \frac{P_{15}\lambda_{13} - P_{16}\lambda_{23}}{2(\lambda_{13}^2 + \lambda_{23}^2)}$, $P'_{16} = \frac{P_{15}\lambda_{23} + P_{16}\lambda_{13}}{2(\lambda_{13}^2 + \lambda_{23}^2)}$ and $P'_{17} = P'_{12} + P'_{13} + P'_{15}$.

Since: $P_1 = d_3^2$, $P_{12} = \frac{d_1^2 + d_2^2}{2}$, $P_3 = 2d_1d_3$, $P_4 = 2d_2d_3$, $P_5 = \frac{d_1^2 - d_2^2}{2}$ and $P_{16} = d_1d_2$.

$$\langle [N_1(t)]^2 \rangle = Q'_{11} + \frac{1}{T}[Q'_{12}e^{2\lambda_{13}T} + Q'_{13}\cos\lambda_{23}Te^{2\lambda_{13}T} + Q'_{14}\sin\lambda_{23}Te^{\lambda_{13}T} +$$
$$Q'_{15}\cos 2\lambda_{23}Te^{2\lambda_{13}T} + Q'_{16}\sin 2\lambda_{23}T^2e^{2\lambda_{13}T} - Q'_{17}] \quad \textbf{(4.14)}$$

where: $Q'_{11} = Q_{11}$, $Q'_{12} = \frac{Q_{12}}{2\lambda_{13}}$, $Q'_{12} = \frac{Q_{13}\lambda_{13} - Q_{14}\lambda_{23}}{\lambda_{13}^2 + \lambda_{23}^2}$,

$Q'_{14} = \frac{Q_{13}\lambda_{23} + Q_{14}\lambda_{13}}{\lambda_{13}^2 + \lambda_{23}^2}$, $Q'_{15} = \frac{Q_{15}\lambda_{13} - Q_{16}\lambda_{23}}{2(\lambda_{13}^2 + \lambda_{23}^2)}$, $Q'_{16} = \frac{Q_{15}\lambda_{23} + Q_{16}\lambda_{13}}{2(\lambda_{13}^2 + \lambda_{23}^2)}$,

and $Q'_{17} = Q'_{12} + Q'_{13} + Q'_{15}$.

since: $Q'_{11} = D_3^2$, $Q'_{12} = \frac{D_{13}^2 + D_{23}^2}{2}$, $Q'_{13} = 2D_{13}D_3$, $Q'_{14} = 2D_{23}D_3$,

$Q'_{15} = \frac{D_{13}^2 - D_{23}^2}{2}$ and $Q'_{16} = D_{13}D_{23}$.

$$\langle N_1(t)N_2(t) \rangle = I_1 + \frac{1}{T}[I_2e^{2\lambda_{13}T} + I_3\cos\lambda_{23}Te^{2\lambda_{13}T} + I_4\sin\lambda_{23}Te^{\lambda_{13}T} + \quad \textbf{(4.15)}$$
$$I_5\cos 2\lambda_{23}Te^{\lambda_{13}T} + I_6\sin 2Te^{2\lambda_{13}T} - I_{27}]$$

where: $I_1 = H_1$, $I_2 = \frac{H_2}{2\lambda_{13}}$, $I_3 = \frac{H_3\lambda_{13} - H_4\lambda_{23}}{\lambda_{13}^2 + \lambda_{23}^2}$, $I_4 = \frac{H_3\lambda_{23} + H_4\lambda_{13}}{\lambda_{13}^2 + \lambda_{23}^2}$,

$I_5 = \frac{H_5\lambda_{13} - H_6\lambda_{23}}{2(\lambda_{13}^2 + \lambda_{23}^2)}$, $I_6 = \frac{H_5\lambda_{23} + H_6\lambda_{13}}{2(\lambda_{13}^2 + \lambda_{23}^2)}$, and $I_7 = I_2 + I_3 + I_5$.

since: $H_1 = d_3D_3$, $H_2 = \frac{d_1D_{13} + d_2D_{23}}{2}$, $H_3 = d_1D_3 + d_3D_{13}$, $H_4 = d_2D_3 + d_3D_{23}$,

$H_5 = \frac{d_1D_{13} - d_2D_{23}}{2}$, and $H_6 = \frac{d_1D_{23} + d_2D_{13}}{2}$.

From definition in [1, 2] we can calculation $G^{(2)}(t)$ for $N_1(t)$, $N_2(t)$ and calculation $C^{(2)}(t)$ for $[N_1(t)]^2$, $[N_2(t)]^2$ and $N_1(t)\,N_2(t)$.

CONCLUSIONS

This paper is devoted for studying a simple model of single species of biological system, where we can get the solutions of the governing equations. And then calculate the correlation functions related to the solutions. After that we studied the evolution of the lotka-volterra interacting model, then we got the solutions of non-linear system by approximated method, and evaluate the correlation functions.

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