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USED EIGENVALUES AND EIGENVECTORS TO COMPRESS THE IMAGE

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Abstract:-
The article's main aim is to point out the significant applications of the linear algebra in the medical engineering field. Hence, the eigenvectors and eigenvalues which represent the core of linear algebra are discussed in details in order to show how they can be used in many engineering applications. The principal components analysis is one of the most important compression and feature extraction algorithms used in the engineering field (1). It mainly depends on the calculation and extraction of eigenvalues and eigenvectors that then used to represent an input; whether it's the image or a simple matrix. In this article, the use of principal component analysis for medical image compression is an important and novel application of linear algebra (2).

Keywords: - *Linear algebra; image processing; eigenvectors; eigenvalues; principal components analysis; Compression.*

INTRODUCTION TO EIGENVALUES AND EIGENVECTORS

If we multiply an n x n matrix by an n x 1 vector we will get a new n x 1 vector back. In other words,

$$A\mu = Y \qquad (1)$$

What we want to know is if it is possible for the following to happen. Instead of just getting a brand new vector out of the multiplication is it possible instead to get the following,

$$A\mu = \lambda\mu$$

In other words is it possible, at least for certain λ and μ, to have matrix multiplication be the same as just multiplying the vector by a constant. So, it is possible for this to happen, however, it won't happen for just any value of λ or μ. If we do happen to have a λ and μ for which this works (and they will always come in pairs) then we call λ and **eigenvalue** of A and μ an **eigenvector** of A (3).

So, how do we go about find the eigenvalues and eigenvectors for a matrix? We first notice that if $\mu = 0$ then (1) is going to be true for any value of λ and so we are going to make the assumption that $\mu \neq 0$. With that out of the way let's rewrite (1) a little.

$$\left.\begin{aligned} A\mu - \lambda\mu &= 0 \\ A\mu - \lambda I_n\mu &= 0 \\ (A - \lambda I_n)\mu &= 0 \end{aligned}\right\} \qquad (2)$$

Notice that before we factored out the μ we added in the appropriately sized identity matrix. This is equivalent to multiplying things by a one and so doesn't change the value of anything. We needed to do this because without it we would have had the difference of a matrix, a, and a constant, λ, and this can't be done. We now have the difference of two matrices of the same size which can be done (4).

So, with this rewrite we see that

$$(A - \lambda I_n)\mu = 0 \qquad (3)$$

In order to find the eigenvectors for a matrix we will need to solve a homogeneous system. We will either have exactly one solution $\mu = 0$ or we will have infinitely many nonzero solutions. Since we've already said that don't want $\mu = 0$ this means that we want the second case.

Knowing this will allow us to find the eigenvalues for a matrix. We will need to determine the values of λ for which we **get,**

$$det(A - \lambda I) = 0$$

Once we have the eigenvalues we can then go back and determine the eigenvectors for each eigenvalue. Let's take a look at a couple of quick facts about eigenvalues and eigenvectors (5).

Fact

If A is an n x n matrix then $det(A - \lambda I) = 0$ is an n^{th} degree polynomial. This polynomial is called the characteristic polynomial .To find eigenvalues of a matrix all we need to do is solve a polynomial. That's generally not too bad provided we keep n small. Likewise this fact also tells us that for an n x n matrix, A, we will have n eigenvalues if we include all repeated eigenvalues (6).

PCA based eigenvectors and eigenvalues

Principal Components Analysis (PCA) is a way of identifying patterns in data, and expressing the data in such a way as to highlight their similarities and differences. It is one of several statistical tools available for reducing the dimensionality of a data set based on calculating eigenvectors and eigenvalues of the input data. Since patterns in data can be hard to find in data of high dimension, where the luxury of graphical representation is not available, PCA is a powerful tool for analyzing data. The other main advantage of PCA is that once you have found these patterns in the data, and you compress the data, i.e. by reducing the number of dimensions, without much loss of information. This technique used in image compression, as we will see in a later. This will take you through the steps you needed to perform a Principal Components Analysis on a set of data (7).

Definition

Let X_{jk} indicate the particular value of the k^{th} variable that is observed on the j^{th} item. We let n be the number of items being observed and p the number of variables measured. Such data are organized and represented by a rectangular matrix X given by a multivariate data matrix.

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{np} \end{bmatrix}$$

In a single-variable case where the matrix X is n x1,

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad (4)$$

The mean

$$\overline{x} = \frac{1}{n}\sum_{j=1}^{n} x_j \quad (5)$$

And the variance

$$s^2 = \frac{1}{n}\sum_{j=1}^{n}(x_j - \overline{x})^2 \quad (6)$$

In addition, the square root of the sample variance is known as the sample standard deviation.
Mean of the k^{th} variable

$$\overline{x_k} = \frac{1}{n}\sum_{j=1}^{n} x_{jk}, \qquad k = 1,2,3,\ldots.\ p \quad (7)$$

Variance of the k^{th}variable

$$s_k^2 = \frac{1}{n}\sum_{j=1}^{n}(x_{jk} - \overline{x_k})^2, \qquad k = 1,2,3,\ldots.\ p \quad (8)$$

For convenience of matrix notation, we shall use the alternative notation S_{kk} for the variance of the k^{th} variable; that is,

$$k^{th} = \frac{1}{n}\sum_{j=1}^{n}(x_{jk} - \overline{x_k})^2, \qquad k = 1,2,3,\ldots.\ p \quad (9)$$

A measure of the linear association between a pair of variables is provided by the notion of covariance. The measure of association between the i^{th} and k^{th} variables in the multivariate data matrix X is given by

$$i^{th} = \frac{1}{n}\sum_{j=1}^{n}(x_{ij} - \overline{x_i})(x_{jk} - \overline{x_k}), \qquad k = 1,2,3,\ldots.\ p,\ k = 1,2,3,\ldots.\ p \quad (10)$$

Which is the average product of the deviations from their respective means. It follows that sjk = s_{ki}, for all i and k, and that for $i = k$, the covariance is just the variance, $s^2_k = s_{kk}$ Matrix of variances and covariance

$$S_n = \begin{bmatrix} s_{11} & \cdots & s_{1p} \\ \vdots & \ddots & \vdots \\ s_{n1} & \cdots & s_{np} \end{bmatrix} \quad (12)$$

The matrix S_n is a symmetric matrix whose diagonal entries arc the sample variances and the subscript n is a notational device to remind us that the divisor n was used to compute the variances and covariance. The matrix S_n is called the covariance matrix.

Theorem 1
Let S_n be the p x p covariance matrix associated with the multivariate data matrix X.
Let the eigenvalues of S_n be λ_j,j = 1, 2,….,p $\lambda_1 \geq \lambda_2 \geq \ldots\ldots \geq \lambda_p \geq 0$, and let the associated orthonormal eigenvectors be $\mathbf{u}_j$, j = 1, 2 ....., p. Then the i^{th} principal component $\mathbf{y}_i$ is given by the linear combination of the columns of **X**, where the coefficients are the entries of the eigenvector $\mathbf{u}_i$; that is, $y_i = i^{th}$ principal component = Xu_i

PCA for image compression
Principal Component Analysis (PCA) was used for the recognition of patterns and compression of digital images used in Medicine. The description of Principal Component Analysis is made by means of the explanation of eigenvalues and eigenvectors of a matrix. This concept is presented on a digital image collected in the clinical routine of a hospital, based

on the functional aspects of a matrix. The analysis of potential for recovery of the original image was made in terms of the rate of compression obtained.

Principal Components Analysis (PCA) is a mathematical formulation used in the reduction of data dimensions. Thus, the PCA technique allows the identification of standards in data and their expression in such a way that their similarities and differences are emphasized. Once patterns are found, they can be compressed, i.e., their dimensions can be reduced without much loss of information (8).

MRI (Magnetic Resonance Imaging) image compression using PCA

The steps normally followed in a PCA of a digital image can now be established:

Step 1: In the computational model of a digital image, in equation (10), the variables X1, X2,...,Xp are the columns of the image. The PCA is begun by coding (correcting) the image to that its columns have zero means and unitary variances. This is common, in order to avoid one or the other of the columns having undue influence on the principal components (9):

Image corrected by the mean = image – mean of the image

Step 2: The covariance matrix C is calculated using equation (11), implemented computationally, that is:

CovImage = image corrected by the mean × (image corrected by the mean)^T

Step 3: The eigenvalues l_1, l2 ,...,l_p and the corresponding eigenvectors a_1 , a_2 ,..., a_p. are calculated.

Step 4: The value of a vector of characteristics is obtained, a matrix with vectors containing the list of eigenvectors (matrix columns) of the covariance matrix vc = (av_1, av_2 , av_3 ,..., av_n)

Step 5: The final data are obtained, that is, a matrix with all the eigenvectors (components) of the covariance matrix.

Final data = vc^T × (Image - mean)^T

Step 6: The original image is obtained from the final data without compression using the expression Image T = (vc)^T × final data + mean^T

Step 7: Any components that explain only a small portion of the variation in data for the effect of image compression are discarded. The eliminations have the effect of reducing the quantity of eigenvectors of the characteristics vectors and can produce final data with a smaller dimension.

Compression ration

The low-loss compression afforded by the present method may be expressed in terms of the compression factor of (r) and of the mean squared error (MSE) committed in the approximation of A (original image) by Ã (image obtained from the disposal of some of the components) (9). The compression factor is defined by:

$$\rho = \frac{\textit{Unit of memory required to represent } \tilde{A}}{\textit{Unit of memory required to represent } A} \quad (12)$$

And the MSE committed in the approximation of A by Ã is:

$$MSE = \frac{1}{L-1}\sum_{i=0}^{L-1}(a_i - \overline{a_i})_2 \quad (13)$$

Example 1

Recovering a TIFF image with 512*512 pixels with all the components (512) of image covariance matrix (without compression, i.e., steps 1 to 6).

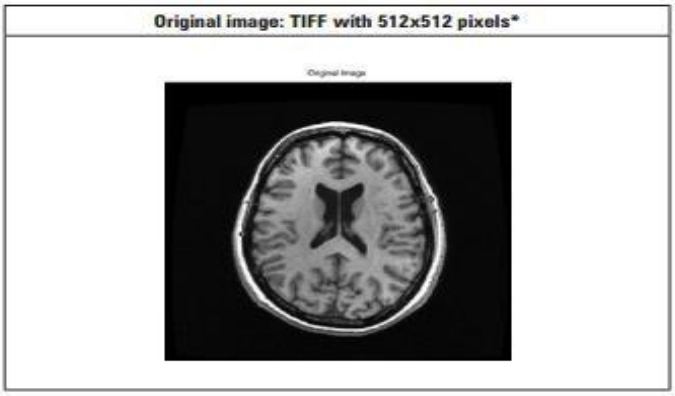

Figure 1-a: MRI original image (512*512)

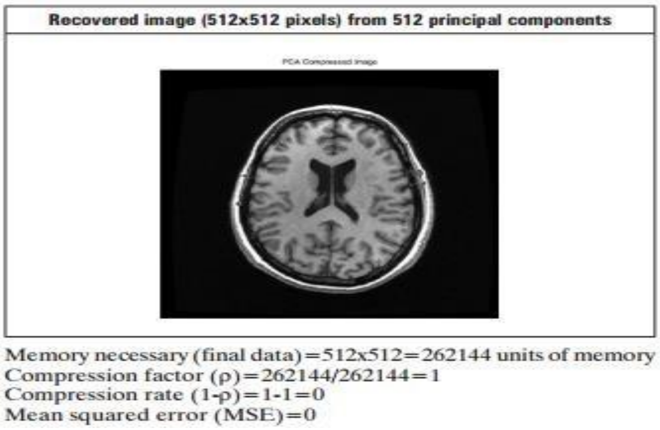

Figure 1-b: Recovered image without compression

Example 2

Recovery of an image with 32 principal components of the image covariance matrix (with compression).

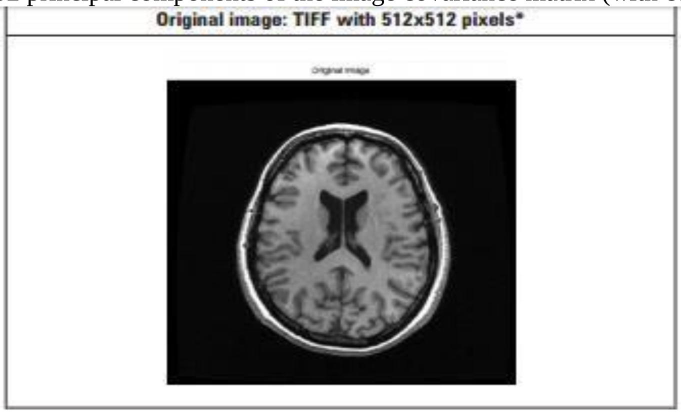

Figure 2-a: Original MRI image 3

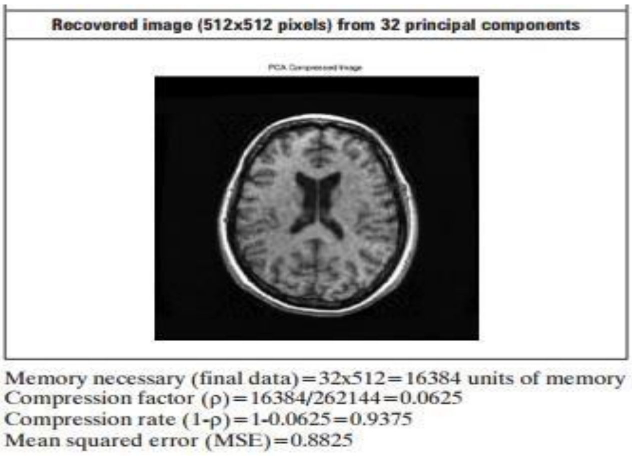

Figure2-b: Compressed MRI image (32*512)

RESULTS DISCUSSION

The Examples show the effects of the reduction in number of principal components (elevation of the image compression rate) in the increased loss of information. This application may bring great savings in storage of medical images. However, the level of information preserved depends on the parameters (compression rate), and should be modulated by the user's interest. The higher the compression rate (the fewer principal components are used in the characteristics vector) the more degraded the quality of the image recovered. In certain applications, such as brain function images, the central principle is the variation of the resonance signal over time. In these conditions, the spatial information may be maintained in a reference file, making it possible to compress subsequent images with no loss.

On the other hand, it is still necessary to evaluate the pertinence of the application of high compression rates when an assessment of structures of reduced dimensions relative to the size of the voxels is needed. Furthermore, the observation of the results from the application of the PCA technique in medical images may be considered a complexity measure.

In other words, images with dense texture patterns tend to produce different results with the use of the technique described. Only points to the line of investigation, in which the results may certify and quantify this possibility. New secondary applications (based on the results here described) may encompass various conditions in the medical routine.

CONCLUSION

Overall, in addition to its mathematical usages, linear algebra has broad usages and applications in most of engineering, medical, and biological field. As science and engineering disciplines grow so the use of mathematics grows as new mathematical problems are encountered and new mathematical skills are required. In this respect, linear algebra has been particularly responsive to computer science as linear algebra plays a significant role in many important computer science undertakings.

The broad utility of linear algebra to computer science reflects the deep connection that exists between the discrete nature of matrix mathematics and digital technology. In this paper we have seen one important applications of the linear algebra which is called principal components analysis. This technique is used broadly in the medical field for compressing the medical images while keeping the good and needed features. However, this is not the only application of linear algebra in this field. Linear algebra has many other applications in this field. It provides many other concepts that are crucial to many areas of computer science, including graphics, image processing, cryptography, machine learning, computer vision, optimization, graph algorithms, quantum computation, computational biology, information retrieval and web search. Among these applications are face morphing, face detection, image transformations such as blurring and edge detection, image perspective removal, classification of tumors as malignant or benign, integer factorization, error-correcting codes, and secret-sharing.

REFERENCES

[1].Santo, R.D. (2012).Principal Component Analysis Applied to Digital Image Compression. Einstein, 10(2), 135-140.

[2].Rajendra, B. (1996). Matrix Analysis, Graduate Texts in Mathematics. Germany: Springer, 158, 35-39.

[3].Jolliffe, I.T. (1986). Principal Component Analysis. New York, NY: SpringerVerlag, 171, 225-229.

[4].Janardan. (2004). Practical Linear Algebra: A Geometry Toolbox. United Kingdom: AK Peters, 73, 34–41.

[5].David, P. (2010). Linear Algebra: A Modern Introduction. United Kingdom: Cengage – Brooks/Cole, 168, 335-339.

[6].Mashal, N., Faust, M., & Hendler, T. (2005). The Role Of The Right Hemisphere In Processing Nonsalient Metaphorical Meanings: Application of Principal Components Analysis To fMRIData. Neuropsychologia, 43(14), 284-300.

[7].Katta, G. (2014). Computational and Algorithmic Linear Algebra and nDimensional Geometry. USA: World Scientific Publishing, 4, 103–107.

[8].Larry, S. (1998). Linear Algebra, Undergraduate Texts in Mathematics. Germany: Springer, 92, 15, 447–453.

[9].Gonzalez, R.C., & Woods, R.E. (1992). Digital Image Processing. Massachusetts: Addison- Wesley, 23, 205-209.