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THE POWER OF SIMPLE CONTINUED FRACTIONS

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Abstract:-

The finite $\alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \ddots + \cfrac{1}{\alpha_n}}} = [\alpha_0; \alpha_1, \ldots, \alpha_n]$ and infinite $\alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \cdots}}}$

Simple continued fractions for (rational and irrational) are considered. The power of the simple continued fractions $[\alpha_0; \alpha_1, \ldots, \alpha_m]$ *are discovered. On other hand we discover how to calculate* $[\alpha_0; \alpha_1, \ldots, \alpha_m]$ *as a simple continued fractions. The most important that we did in this paper, we prove by theorem any two simple continued fractions* $[\alpha_0; \alpha_1, \ldots, \alpha_m] \neq [\beta_0; \beta_1, \ldots, \beta_n]$ *the fraction is* $[\alpha_0; \alpha_1, \ldots, \alpha_m] > [\beta_0; \beta_1, \ldots, \beta_n]$ *or* $[\alpha_0; \alpha_1, \ldots, \alpha_m] < [\beta_0; \beta_1, \ldots, \beta_n]$ *(for any n, m). Many definitions and examples that we used of that low and theorem are presented.*

Key words:-*Simple continued fractions. Multiplicative. Multiplicative inverse, Division*.

INTRODUCTION

There are many applications of continued fractions: combine *continued fractions* with the concepts of *golden ratio* and *Fibonacci numbers,* Pell equations and calculation of fundamental units in quadratic fields, reduction of quadratic forms and calculation of class numbers of imaginary quadratic field [1]. There is a pleasant connection between Chebyshev polynomials, the Pell equation and continued fractions, the latter two being understood to take place in real quadratic function fields rather than the classical case of real quadratic number fields [2]. The simple continued fractions have been studied in mathematical (Diophantine Equation, congruence $ax \equiv b \ (mod\ m)$ and Pell's equations) and physical (gear ratio) [3]. The analytic of continued fractions for the real and complex values have been studied in [4, 5]. However, [3, 6, 7] studied the continued fractions for the integer values. There are many applications of simple continued fractions (Gosper's batting average problem [8], Cryptography …). In [5, 9] we showed that, any number, rational or irrational, can be expression as a finite or infinite continued fraction. Also, the simple continued fractions are useful to solve Diophantine Equation or congruence $ax \equiv b \ (mod\ m)$. The most important, that we did in [7, 9], we defined the addition and subtraction of the simple continued fractions. Also, we showed that, how can we know which simple continued fraction is greater than of the other. Also, we defined the multiplication and division of the simple continued fractions and we explained multiply the numbers $\sqrt{a} \cdot \sqrt{\ } b$ by using the continued fractions. In this paper we defined the power. This study start with some definitions and theorems that we used to defined the multiplication of two simple continued fractions.

The Power of Simple Continued Fractions

Definition 1

An expression of the form

$$\alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \ddots + \cfrac{1}{\alpha_n}}}$$

is called a finite simple continued fraction, where a_n is positive integer for all $n \geq 1$, α_0 can be any integer number. The above fraction is sometime represented by $[\alpha_0;\alpha_1,\alpha_2,\ldots,\alpha_n]$ for finite simple continued fraction and $[\alpha_0;\alpha_1,\alpha_2,\ldots]$ for infinite simple continued fraction. In this paper we will use the symbol (S.C.F.) for the simple continued fraction.

Theorem 1

A number is rational if and only if it can be expressed as a finite S.C.F. [3].

For example

$$\frac{62}{23} = 2+\frac{16}{23} = 2+\cfrac{1}{\frac{23}{16}} = 2+\cfrac{1}{1+\frac{7}{16}} = 2+\cfrac{1}{1+\cfrac{1}{\frac{16}{7}}}$$

$$= 2+\cfrac{1}{1+\cfrac{1}{2+\frac{2}{7}}} = 2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\frac{1}{2}}}} = [2;1,2,3,2].$$

Let $a=\alpha_0$ be an irrational number and define the sequence $a_0, a_1, a_2, \ldots$ recursively by $a_k = \lfloor \alpha_k \rfloor$, $\alpha_{k+1} = \dfrac{1}{\alpha_k - a_k}$ for $k = 0,1,2,\ldots$.

Then α is the values of infinite S.C.F. [$a_0;a_1,a_2,\cdots$] Where, α is the greatest integer number. For example $\sqrt{3} = [a_0;a_1,a_2,a_3,a_4,a_5,\ldots] = [1;1,2,1,2\ldots] = [1;\overline{1,2}]$. We can use the same operations of finite S.C.F. for infinite S.C.F.

Definition 2: The S.C.F. $[\alpha_0;\alpha_1,\ldots,\alpha_n]$ can be defined by

$$[\alpha_0;\alpha_1,\ldots,\alpha_n] = \alpha_0 + \frac{K_{n-1}(\alpha_2)}{K_n(\alpha_1)}, \text{ or } \quad [\alpha_0;\alpha_1,\ldots,\alpha_n] = \frac{K_{n+1}(\alpha_0)}{K_n(\alpha_1)}.$$

Where

$$\begin{aligned}
&K_0(\alpha_0)=1 && K_0(\alpha_1)=1\\
&K_1(\alpha_0)=\alpha_0 && K_1(\alpha_1)=\alpha_1\\
&K_2(\alpha_0)=\alpha_0\alpha_1+1 && K_2(\alpha_1)=\alpha_1\ \alpha_2+1\\
&K_3(\alpha_0)=\alpha_0\alpha_1\alpha_2+\alpha_0+\alpha_2 && K_3(\alpha_1)=\alpha_1\alpha_2\alpha_3+\alpha_1+\alpha_3\\
&\quad\vdots && \quad\vdots\\
&K_i(\alpha_0)=\alpha_{i-1}\ K_{i-1}(\alpha_0)+K_{i-2}(\alpha_0) && K_i(\alpha_1)=\alpha_i\ K_{i-1}(\alpha_1)+K_{i-2}(\alpha_1)
\end{aligned}$$

In general

$$K_i(\alpha_j) = \alpha_{i+(j-1)} K_{i-1}(\alpha_j) + K_{i-2}(\alpha_j) \quad , \qquad i = 1,2,\ldots,n \quad , \quad j = 0,1,\ldots,n$$

$$K_{-i}(\alpha_j) = 0 \qquad\qquad K_0(\alpha_j) = 1 .$$

Lemma 1

If $c_j = 0$ in $[c_0; c_1, \ldots, c_{j-1}, c_j, c_{j+1}, \ldots, c_n]$, for some j with $0 < j < n$, then we can replace c_{j-1} by $c_{j-1} = c_{j-1} + c_{j+1}$ and delete c_j, c_{j+} From the simple continued fraction expansion, without changing the value of S.C.F. [5].

Lemma 2

If $c_j = c_{j+1} = 0$ in $[c_0; c_1, \ldots, c_{j-1}, c_j, c_{j+1}, c_{j+2}, \ldots, c_n]$, for some j with $0 < j < n$ we can delete c_j, c_{j+1} From the simple continued fraction expansion, without changing the value of S.C.F. [5].

Definition 3 Let $[\alpha_0; \alpha_1, \ldots, \alpha_m]$ be S.C.F. , we define $[\alpha_0; \alpha_1, \ldots, \alpha_m]^n$ by

$$[\alpha_0; \alpha_1, \ldots, \alpha_m]^n = [s_0; s_1, \ldots, s_m] \qquad (1)$$

where

$s_0 = \alpha_0^n$

$$s_1 = \left\lfloor \frac{K_1^n(\alpha_1)}{K_2^n(\alpha_0) - s_0 K_1^n(\alpha_1)} \right\rfloor,$$

$$s_2 = \left\lfloor \frac{K_0(s_1) K_3^n(\alpha_0) - K_1(s_0) K_2^n(\alpha_1)}{K_2(s_0) K_2^n(\alpha_1) - K_1(s_1) K_3^n(\alpha_0)} \right\rfloor,$$

$$s_3 = \left\lfloor \frac{K_2(s_0) K_3^n(\alpha_1) - K_1(s_1) K_4^n(\alpha_0)}{K_2(s_1) K_4^n(\alpha_0) - K_3(s_0) K_3^n(\alpha_1)} \right\rfloor,$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$s_i = \begin{cases} \left\lfloor \dfrac{K_{i-1}(s_0) K_i^n(\alpha_1) - K_{i-2}(s_1) K_{i+1}^n(\alpha_0)}{K_{i-1}(s_1) K_{i+1}^n(\alpha_0) - K_i(s_0) K_i^n(\alpha_1)} \right\rfloor, & i \text{ odd} \\[2ex] \left\lfloor \dfrac{K_{i-2}(s_1) K_{i+1}^n(\alpha_0) - K_{i-1}(s_0) K_i^n(\alpha_1)}{K_i(s_0) K_i^n(\alpha_1) - K_{i-1}(s_1) K_{i+1}^n(\alpha_0)} \right\rfloor, & i \text{ even} \end{cases} \qquad \text{for } i = 1,2,\ldots, m.$$

The last term s_m of the resulting S.C.F. is to be expanded again as a S.C.F. if necessary and not to be treated as greatest integer number as have the preceding terms been treated.

Example 1: Find $[2; 1, 4]^{2}$.

Solution: Let [2;1, 4] = $[\alpha_0; \alpha_1, \alpha_2]$, we get $m = 2$, $n = 2$, from equation (1) we have

$$[2;1,4]^2 = [s_0; s_1, s_2]$$

Where s_2 is the last term

$s_0 = \alpha_0^2 = (2)^2 = 4$,

$$s_1 = \left\lfloor \frac{K_1^2(\alpha_1)}{K_2^2(\alpha_0) - s_0 K_1^2(\alpha_1)} \right\rfloor = \left\lfloor \frac{(\alpha_1)^2}{(\alpha_0\alpha_1+1)^2 - s_0(\alpha_1)^2} \right\rfloor$$

$$= \left\lfloor \frac{(1)^2}{(2\cdot 1+1)^2 - 4\cdot(1)^2} \right\rfloor = \left\lfloor \frac{1}{5} \right\rfloor = 0,$$

$$s_2 = \frac{K_0(s_1)K_3^2(\alpha_0) - K_1(s_0)K_2^2(\alpha_1)}{K_2(s_0)K_2^2(\alpha_1) - K_1(s_1)K_3^2(\alpha_0)}$$

$$= \frac{(\alpha_0\alpha_1\alpha_2 + \alpha_0 + \alpha_2)^2 - s_0\cdot(\alpha_1\alpha_2+1)^2}{(s_0 s_1 + 1)(\alpha_1\alpha_2+1)^2 - s_1\cdot(\alpha_0\alpha_1\alpha_2+\alpha_0+\alpha_2)^2}$$

$$= \frac{(2\cdot 1\cdot 4 + 2 + 4)^2 - 4\cdot(1\cdot 4+1)^2}{(4\cdot 0+1)(1\cdot 4+1)^2 - 0\cdot(2\cdot 1\cdot 4+2+4)^2},$$

$$= \frac{96}{25} \text{ (to be treated as S.C.F.)}$$

$$= [3;1,5,4]$$

Therefore

$$[2;1,4]^2 = [4;0,3,1,5,4]$$

$$= [7;1,5,4] \qquad \text{(by lemma 1)}$$

Example 2: Find $[1;2]^5$.

Let $[1;2] = [\alpha_0;\alpha_1]$, we get $m = 1$, $n = 5$, from equation (1), we have

$$[1;2]^5 = [s_0;s_1],$$

Where s_1 is the last term and

$$s_0 = \alpha_0^5 = (1)^5 = 1$$

$$s_1 = \frac{K_1^5(\alpha_1)}{K_2^5(\alpha_0) - s_0 K_1^5(\alpha_1)} = \frac{(\alpha_1)^5}{(\alpha_0\alpha_1+1)^5 - s_0(\alpha_1)^5}$$

$$= \frac{(2)^5}{(1\cdot 2+1)^5 - 1\cdot(2)^5},$$

$$= \frac{32}{211} \qquad \text{(to be treated as S.C.F.)}$$

$$= [0;6,1,1,2,6]$$

Therefore,

$$[1;2]^5 = [1;0,6,1,1,2,6]$$

$$= [7;1,1,2,6] \qquad \text{(by lemma 1)}$$

Example 3: Find $[1;\overline{1,2}]^2$.

Solution:

Let $[1;\overline{1,2}] = [1;1,2,1,2,\ldots] = [\alpha_0;\alpha_1,\alpha_2,\alpha_3,\alpha_4,\ldots]$

The results get: $n = 2$, from equation (1) we have

$$[1;\overline{1,2}]^2 = [s_0;s_1,s_2,\ldots]$$

Where

$$s_0 = \alpha_0^2 = (1)^2 = 1,$$

$$s_1 = \left\lfloor \frac{K_1^2(\alpha_1)}{K_2^2(\alpha_0) - s_0 K_1^2(\alpha_1)} \right\rfloor$$
$$= \left\lfloor \frac{1}{4-1} \right\rfloor = \left\lfloor \frac{1}{3} \right\rfloor = 0,$$

$$s_2 = \left\lfloor \frac{K_0(s_1)K_3^2(\alpha_0) - K_1(s_0)K_2^2(\alpha_1)}{K_2(s_0)K_2^2(\alpha_1) - K_1(s_1)K_3^2(\alpha_0)} \right\rfloor$$
$$= \left\lfloor \frac{25-9}{9} \right\rfloor = 1,$$

$$s_3 = \left\lfloor \frac{K_2(s_0)K_3^2(\alpha_1)}{K_4^2(\alpha_0) - K_3(s_0)K_3^2(\alpha_1)} \right\rfloor$$
$$= \left\lfloor \frac{16}{49-2(16)} \right\rfloor = 0,$$

$$s_4 = \left\lfloor \frac{K_2(s_0)K_3^2(\alpha_1)}{K_4^2(\alpha_0) - K_3(s_0)K_3^2(\alpha_1)} \right\rfloor$$
$$= \left\lfloor \frac{16}{49-2(16)} \right\rfloor = 0,$$

$$s_5 = \left\lfloor \frac{K_4(s_0)K_5^2(\alpha_1)}{K_4(s_1)K_6^2(\alpha_0) - K_5(s_0)K_5^2(\alpha_1)} \right\rfloor$$
$$= \left\lfloor \frac{(11+4)^2}{(19+7)^2 - 2(15)^2} \right\rfloor$$
$$= \left\lfloor \frac{225}{226} \right\rfloor = 0,$$

$$s_6 = \left\lfloor \frac{K_4(s_1)K_7^2(\alpha_0) - K_5(s_0)K_6^2(\alpha_1)}{K_6(s_0)K_6^2(\alpha_1) - K_5(s_1)K_7^2(\alpha_0)} \right\rfloor$$
$$= \left\lfloor \frac{(52+19)^2 - 2(10+11)^2}{(41)^2} \right\rfloor$$
$$= \left\lfloor \frac{1679}{1681} \right\rfloor = 0,$$

Therefore,

$$s_i = \left\lfloor \frac{a}{a+1} \right\rfloor \qquad (a \text{ is integer, } i \text{ (odd)})$$

if $i \to \infty$ then $a \to \infty$ and $s_i \to 1$,

therefore

$$[1;\overline{1,2}]^2 = [1;0,1,0,0,\ldots,1]$$
$$= [2;0,0,\ldots,1] \qquad \text{(by lemma 1)}$$
$$= 3 \qquad \text{(by lemma 2)}$$

Theorem 3

Let $[a_0;a_1,\ldots,a_m]$ and $[b_0;b_1,\ldots,b_n]$ be two S.C.F. with $a_i \neq 0$ for all $i = 1,2,\ldots,m$ and, $b_j \neq 0$ for all $j = 1,2,\ldots,n$ and $[a_0;a_1,\ldots,a_m] \neq [b_0;b_1,\ldots,b_n]$ then:

(i) If $a_0 \neq b_0$, suppose $a_0 > b_0$, then

$$[a_0;a_1,\ldots,a_m] > [b_0;b_1,\ldots,b_n].$$

(ii) If $a_i = b_i$ for all $i = 0,1,2,\ldots k$, $k < \min(m,n)$,

suppose $a_{k+1} > b_{k+1}$, with $k+1 \leq \min(m,n)$ then

$[a_0;a_1,\ldots,a_m] > [b_0;b_1,\ldots,b_n]$ if k is odd and

$[a_0;a_1,\ldots,a_m] < [b_0;b_1,\ldots,b_n]$ if k is even.

(iii) If $a_i = b_i$ for all $i = 1,2,\ldots,k$, $k = \min(m,n)$

Suppose $k = m$, then

$[a_0;a_1,\ldots,a_m] > [b_0;b_1,\ldots,b_n]$ if k is odd

And

$$[a_0;a_1,\dots,a_m] < [b_0;b_1,\dots,b_n] \qquad \text{if } k \text{ is even.}$$

Example

(i) $[3;2,2] > [2;1,4]$ since if $[3;2,2] = [a_0;a_1,a_2]$ and $[2;1,4] = [b_0;b_1,b_2]$, From theorem 3 (i) we have

$$a_0 > b_0,$$

hence

$$[3;2,2] > [2;1,4],$$

(ii) $[3;1,1,3] > [3;7]$ since let $[3;1,1,3] = [a_0;a_1,a_2,a_3]$, $m = 3$ and $[3;7] = [b_0;b_1]$, $n = 1$, from theorem 3 (ii) we have

$a_0 = b_0$, $k = 0$

since $a_1 = 1$, $b_1 = 7$

and $b_1 > a_1$

hence

$[1;2,3] < [1;2,3,4]$ (since k is even)

(iii) $[1;2,3] < [1;2,3,4]$ since let $[1;2,3] = [a_0;a_1,a_2]$, $m = 2$ and $[1;2,3,4] = [b_0;b_1,b_2,b_3]$, $n = 3$, from theorem 3 (iii) we have:

$a_i = b_i$ for all $i = 0,1,2$

then $k = m = 2$ (even)

hence

$[1;2,3] < [1;2,3,4]$ (since k is even)

(iv) $[1;2,3,4] > [1;2,3,4,5]$ since if $[1;2,3,4] = [a_0;a_1,a_2,a_3]$, tthen $m = 3$ and $[1;2,3,4,5] = [b_0;b_1,b_2,b_3,b_4]$, $n = 4$, from theorem 3 (iii) we have

$a_i = b_i$ for all $i = 0,1,2,3$

then $k = m = 3$ (odd)

hence

$[1;2,3,4] > [1;2,3,4,5]$. (since k is odd)

CONCLUSION

This paper is the third part for the operations of the simple continued fractions. In the first and the second parts [6] we discovered the definitions of addition, subtractions, the multiplication, multiplicative inverse and the division of the simple continued fractions. In this paper we discovered the definition of power of simple continued fraction. Finally, we defined the order of the simple continued fractions.

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